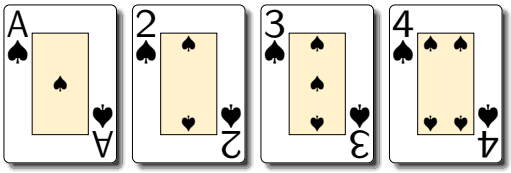
Math 372: Worksheet 3

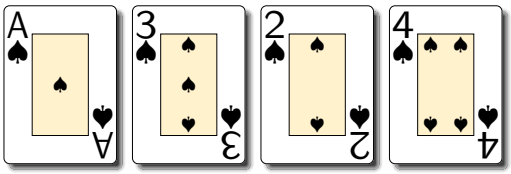
February 20, 2026

Instructions: *Write your answers on a separate sheet.*

Problem 1. Suppose we lay out the following four playing cards in a line like this:



A ***random transposition*** is where two different cards are selected at random and their positions swapped. For example, the transposition (23) swaps cards 2 and 3. If the cards started in the above order, then the effect of applying (23) would be to reorder the cards as follows:



Anna's nemesis Mark challenges her to a betting game. In the game, Mark lays out four cards in the order $1, 2, 3, 4$ and then performs k random transpositions, one after the other. As this happens, Anna is blindfolded, so she can't see which transpositions were performed—however, because she can hear Mark moving the cards, she does know the value of k. Anna then has to guess the final order of the cards. If she guesses correctly, Mark will pay her \$20; otherwise she has to pay Mark \$1.

Mark believes that as long as he performs at least 4 transpositions, then all $4! = 24$ permutations of the cards will be equally likely, so that Anna is expected to lose money when she plays the game. Moreover Mark believes that Anna is a sucker and he can make lots of money from her.

But Mark is wrong. Design a guessing strategy for Anna to bleed Mark for all he's worth.

Hint: *First consider the simpler problem where the game is played with just 2 cards, and design a guessing strategy for Anna in that case.*

Then, consider the game with 3 cards. In that case, there are six outcomes (shown below). For each $k = 2, 3, 4, 5$, write down a table which gives the probability mass function these six outcomes. This will lead you to to a good guessing strategy for Anna.



Once you understand the case with 3 cards, move to 4.

Problem 2. What does Mark think his probability of winning is? What does he think his expected payoff for playing the game is?

Hint: use the formula

$$\text{expected payoff} = (\text{payoff if he wins})\mathbb{P}[\text{win}] + (\text{payoff if he loses})\mathbb{P}[\text{lose}]$$

Problem 3. What is Anna's expected payoff if she adopts your strategy? If she adopts your strategy, how much money can she expect to make if she agrees to play the game 100 times?

Problem 4 (Exercise 3.2.16)**.** For the last 21 months, the Ukrainian city of Kherson has been subject to an ongoing campaign of drone attacks by Russian forces entrenched across the nearby Dnipro River. Suppose that 25% of Ukrainian air-defense crews in the city are equipped with an experimental electromagnetic drone jammer.

Suppose that, on a given night, four crews are selected at random to defend a neighborhood near the riverbank. Let X denote the number among the four that are equipped with the jammer.

(a) Find the probability distribution of X. *[Hint: Let J denote a crew equipped with the jammer and N one who does not. Then one possible outcome is* `JJNJ`*, with probability* $(.25)(.75)(.25)(.25)$ *and associated* X *value* 3*. There are 15 other outcomes.]*

(b) Draw the corresponding probability histogram.

(c) What is the most likely value for X?

(d) What is the probability that at least two of the four selected crews have the jammer?

Problem 5 (Excersise 3.2.21 in the textbook)**.** Suppose that you read through this years issues of the New York Times and record each number that appears in a news article—the income of a CEO, the number of cases of wine produced by a winery, the age of a celebrity, and so on. Now focus on the leading digit of each number, which could be $1, 2, \ldots, 8$, or 9. Your first thought might be that the leading digit X of a randomly selected number would be equally likely to be one of the nine possibilities. However, much empirical evidence as well as some theoretical arguments suggest an alternative probability distribution called Benford's law:

$$p(x) = \mathbb{P}\left[\text{first digit is } x\right] = \log_{10}\left(\frac{x+1}{x}\right), \quad \text{for } x = 1, 2, \ldots, 9.$$

(a) Without computing individual probabilities from this formula, show that it specifies a legitimate pdf (i.e., show that $\sum_{k=1}^{9} p(k) = 1$ and that $p(x) \geq 0$ for all $x = 1, \ldots, 9$).

(b) Now compute the individual probabilities and compare to the corresponding discrete distribution.

(c) Plot the cdf of X.

(d) Using the cdf, what is the probability that the leading digit is at most 3? At least 5?

[Note: Benford's law is the basis for some auditing procedures used to detect fraud in financial reporting—for example, by the Internal Revenue Service]

For the remaining problem, we'll need the following definition.

Definition 6. Let X be a discrete random variable with finite expected value $\mu = \mathbb{E}[X]$. The ***variance*** of X, denoted $\mathrm{Var}(X)$, is the quantity

$$\mathrm{Var}(X) = \sum_{x} (x - \mu)^2$$

where the sum ranges over all distinct values x that the random variable X can take.

Problem 7. A certain brand of upright freezer is available in three different rated capacities: 16, 18, and 20 cubic feet. Let X be the rated capacity of a freezer of this brand sold at a certain store. Suppose that X has probability mass function

x	16	18	20
$p(x)$	.2	.5	.3

(a) Compute $\mathbb{E}[X], \mathbb{E}\left[X^2\right], \mathrm{Var}(X)$.

(b) If the price of a freezer having capacity X is $70X - 650$, what is the expected price paid by the next customer to buy a freezer?

(c) What is the variance of the price paid by the next customer?

(d) Suppose that although the rated capacity of a freezer is X, the actual capacity is $h(X) = X - .008X^2$. What is the expected actual capacity of the freezer purchased by the next customer.