23 2026-03-25 | Week 10 | Lecture 23

23.1 An example of using a joint distribution to compute probabilities

Example 105. Let X and Y be the lifetimes of two lightbulbs, in months. Assume that the expected lifetimes are 10 months and 12 months, respectively. Also assume that the lifetimes are independent, exponentially-distributed random variables.

Then

$$f_X(x) = \frac{1}{10}e^{-\frac{1}{10}x}\mathbf{1}_{[x>0]} \quad \text{and} \quad f_Y(y) = \frac{1}{12}e^{-\frac{1}{12}y}\mathbf{1}_{[y>0]}.$$

Since X and Y are independent, the joint pdf is

$$\begin{aligned} f(x,y) &= f_X(x)f_Y(y) \\ &= \left(\frac{1}{10}e^{-\frac{1}{10}x}\mathbf{1}_{[x>0]}\right)\left(\frac{1}{12}e^{-\frac{1}{12}y}\mathbf{1}_{[y>0]}\right) \\ &= \frac{1}{120}e^{-\frac{1}{10}x-\frac{1}{12}y}\mathbf{1}_{[x,y>0]} \\ &= \begin{cases} \frac{1}{120}e^{-\frac{1}{10}x-\frac{1}{12}y} & : \quad x>0 \text{ and } y>0 \\ 0 & : \quad \text{else} \end{cases} \end{aligned}$$

Question: What is the probability that both lightbulbs last at least 9 months?

Want:

$$\begin{aligned} \mathbb{P}\left[X \geq 9, Y \geq 9\right] &= \int_9^\infty \int_9^\infty f(x,y)dxdy \\ &= \int_9^\infty \int_9^\infty f_X(x)f_Y(y)dxdy \\ &= \int_9^\infty \left[\int_9^\infty f_X(x)f_Y(y)dx\right]dy \\ &= \int_9^\infty \left[\int_9^\infty f_X(x)dx\right]f_Y(y)dy \\ &= \left(\int_9^\infty f_X(x)dx\right)\left(\int_9^\infty f_Y(y)dy\right) \\ &= \left(\int_9^\infty \frac{1}{10}e^{-\frac{1}{10}x}dx\right)\left(\int_9^\infty \frac{1}{12}e^{-\frac{1}{12}y}dy\right) \\ &= \left[-e^{\frac{1}{10}x}\right]_{x=9}^{x=\infty} \cdot \left[-e^{\frac{1}{12}y}\right]_{y=9}^{y=\infty} \\ &= e^{-9\left(\frac{1}{10}+\frac{1}{12}\right)} \\ &= e^{-1.65} \\ &\approx .2. \end{aligned}$$

So with probability about one fifth, both ligthbulbs will last at least 900 hours.

End of Example 105. □

23.2 LOTUS Theorem - multivariate case

Chapter 5.2 in the textbook.

Theorem 106 (LOTUS - multivariate case)**.** *Let X, Y be random variables with joint pmf $p(x,y)$ if X, Y are discrete, or joint pdf $f(x,y)$ if X, Y are continuous. Then for any function $h : \mathbb{R}^2 \to \mathbb{R}$,*

$$\mathbb{E}\left[h(X,Y)\right] = \begin{cases} \sum\limits_{all\ x,y} h(x,y)p(x,y) & : \quad \textit{if } X,Y \textit{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy & : \quad \textit{if } X,Y \textit{ are continuous} \end{cases}$$

An important aplication of Theorem 106:

Theorem 107. *If X, Y are indepenedent random variables then*

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

Proof. Let's prove the case with X, Y discrete. Let $p(x, y)$ be the joint pmf of X and Y. Taking $h(x, y) = xy$, Theorem 106 implies

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_{\text{all } x,y} xyp(x, y) \\
&= \sum_{\text{all } x,y} xyp_X(x) \cdot p_Y(y) && \text{by independence} \\
&= \left(\sum_{\text{all } x} xp_X(x)\right)\left(\sum_{\text{all } y} yp_Y(y)\right) \\
&= \mathbb{E}[X]\,\mathbb{E}[Y] && \text{by Theorem 60}
\end{aligned}$$

The continuous case is similar. □

One important function for Theorem 106 is when $h(X, Y) = (X - \mu_X)(Y - \mu_Y)$, where $\mu_X = \mathbb{E}[X]$ and $\mu_Y = \mathbb{E}[Y]$. That gives us a formula for what is called the ***covariance***:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

By foiling this out, we get

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mu_X\mu_Y$$

Interpetation: covariance is a rough measure of the relationship between the two random variables. By Theorem 107, if X, Y are independent, then

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mu_X\mu_Y = \mathbb{E}[X]\,\mathbb{E}[Y] - \mu_X\mu_Y = 0.$$

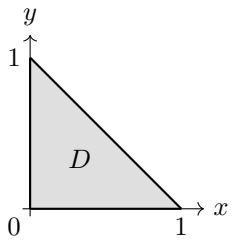
Example 108 (Covariance calculation)**.** Let X, Y be random variables with joint distribution

$$f(x, y) = 24xy$$

whose domain is

$$D := \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\right\}$$

which is shown below:



What is the covariance To compute $\mathbb{E}[XY]$, let's take $h(x, y) = xy$. Recalling that $f(x, y) = 24xy$ for

$(x, y) \in D$, we have

$$\begin{aligned}
\mathbb{E}[XY] &= \int_D h(x,y) f(x,y) dxdy \\
&= \int_D (xy) 24xy dxdy \\
&= 24 \int_D x^2 y^2 dxdy \\
&= 24 \int_0^1 \int_0^{1-y} x^2 y^2 dxdy \\
&= 24 \int_0^1 y^2 \left[\int_0^{1-y} x^2 dx \right] dy \\
&= 8 \int_0^1 y^2 (1-y)^3 dy \\
&= \dots \\
&= -8.
\end{aligned}$$

End of Example 108. □