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Example 88 (Waiting times)**.** Suppose you station yourself at spot on a road and watch as cars pass by. Let T be the time in minutes until the first car passes by. This random variable is called the *waiting time*.

Under certain circumstances, it is reasonable for T to have pdf

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{[x \geq 0]} \tag{13}$$

for a certain (fixed) value of $\lambda > 0$.

For concreteness, let's assume that $\lambda = 3$, so

$$f(x) = 3e^{-3x} \mathbf{1}_{[x \geq 0]}$$

Then the cdf is

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f_T(x) dx \\ &= \int_0^t 3e^{-3x} dx \\ &= 3 \left[\frac{-e^{-4x}}{3} \right]_0^t \\ &= 1 - e^{-3t}. \end{aligned}$$

Equivalently,

$$\mathbb{P}[T > t] = e^{-3t}.$$

For example the probability that you have to wait more than $t = 30$ seconds for a car to pass by is

$$\mathbb{P}[T > 5] = e^{-3 \cdot 0.5} \approx .22$$

What about

$$\mathbb{E}[T]\,?$$

If the cars come at a rate of $\lambda = 3$ cars per minute, then on average how long would you have to wait for a car to arrive? This should be 20 seconds.

End of Example 88. □

Definition 89 (Expected value – continuous random variables)**.** The ***expected value*** of a continuous random variable X is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f(x) dx$$

where $f(x)$ is the pdf of X. (Provided that the integral above converges absolutely).

We also have an analogue of the law of the unconscious statistician (Theorem 60)

Theorem 90 (LOTUS - continuous case)**.** *Let X be a continous random variable with pdf f. Then for any function $h : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) dx,$$

provided that the integral exists and is finite.

Suppose $\mu = \mathbb{E}[X]$. Applying Theorem 90 to the function $h(x) = (x - \mu)^2$ gives the following formula for the variance of a continuous random variable:

$$\mathrm{Var}(X) := \mathbb{E}\left[(X - \mu)^2\right] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx.$$

And again, we have the shortcut formula

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.$$