19 2026-03-06 | Week 08 | Lecture 19

19.1 Continuous random variables

(This lecture is based on sections 4.1-4.3 in the textbook)

Definition 81 (pdf)**.** Let X be a continuous random variable. The ***probability density function*** (aka "pdf" or "density") of X is a function f such that for any real numbers a, b with $a \leq b$, we have

$$\mathbb{P}[X \in (a,b)] = \int_a^b f(x)dx$$

The graph of $y = f(x)$ is called the ***density curve*** of X.

Observatioins about pdfs:

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $f(x)$ is not a probability because it could be greater than 1. Instead, you can think of it vaguely as a "relative likeliood".

Proposition 82. *Let $F(x) = \mathbb{P}[X \leq x]$ be the cdf of X, and let f be the pdf of X. Then*

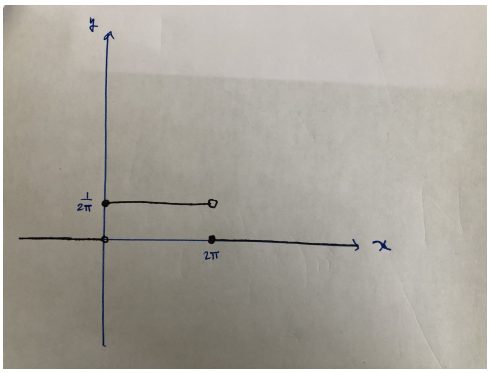
$$F(x) = \int_{-\infty}^{x} f(t)dt \quad \textit{and} \quad F'(x) = f(x)$$

Proposition 83 (Properties of cdfs of uniform random variables)**.**

- $\mathbb{P}[a \leq X \leq b] = F(b) - F(a)$
- *F is a nondecreasing function*
- *$F(-\infty) = 0$ and $F(+\infty) = 1$.*

Example 84 (Pinwheel example continued)**.** Recall the example of the pinwheel Example 79, with the angle X measured in radians, so that the state space of X is $[0, 2\pi)$, with all values equally likely. Recall that we computed the pdf as

$$f_X(x) = \frac{1}{2\pi} \cdot \mathbf{1}_{[0,2\pi)}(x) = \begin{cases} \frac{1}{2\pi} & : \quad x \in [0, 2\pi) \\ 0 & : \quad x \notin [0, 2\pi) \end{cases}$$



Recall also that the cdf is defined as

$$F_X(x) := \int_{-\infty}^{x} f_X(t)dt.$$

So that for the pinwheel angle X, we have

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} \frac{1}{2\pi} \cdot \mathbf{1}_{[0,2\pi)}(t)dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{x} \mathbf{1}_{[0,2\pi)}(t)dt. \end{aligned} \tag{12}$$

To evaluate an integral of this form, we have to consider three cases:

- $x < 0$ (i.e. when x is to the left of the interval)
- $x \geq 2\pi$ (i.e. when x is to the right of the interval)
- $0 \leq x < 2\pi$ (i.e. when x is in the interval)

Let's think through these cases:

- When $x < 0$, the integral on the right-hand side of Eq. (12) is zero.
- When $x \geq 2\pi$, the integral equals

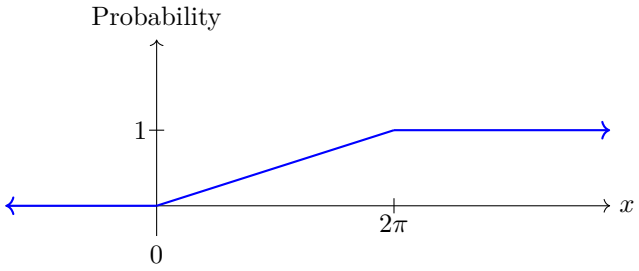
$$\int_{-\infty}^{0} 0dt + \frac{1}{2\pi} \int_{0}^{2\pi} dt + \int_{2\pi}^{x} 0dt = 0 + 1 + 0 = 1$$

- When $0 \leq x < 2\pi$, the integral equals

$$\int_{-\infty}^{0} 0dt + \frac{1}{2\pi} \int_{0}^{x} dt = 0 + \frac{x}{2\pi} = \frac{x}{2\pi}$$

From this analysis and Eq. (12), we can conclude that

$$F_X(x) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x}{2\pi} & : \quad 0 \leq x < 2\pi \\ 1 & : \quad x \geq 2\pi \end{cases}$$



End of Example 84. □

Definition 85 (Uniform random variable)**.** Let I be an interval with endpoints a and b, such that $a < b$. A continuous random variable X has ***uniform distribution*** on I if it has pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & : \quad x \in I \\ 0 & : \quad x \notin I \end{cases}$$

In this case, X is called a ***uniform*** random variable.

Definition 86 (Exponential random variable)**.** We say that X is an ***exponential*** random variable with rate $\lambda > 0$ if it has pdf

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{[x \geq 0]}.$$

Remark 87. In Definition 86, the term $\mathbf{1}_{[x \geq 0]}$ is an "indicator function" which takes value 1 when $x \geq 0$ and 0 if $x < 0$.