18 2026-03-04 | Week 08 | Lecture 18

18.1 Applications of the Poisson random variable

(Based on section 3.6 in the textbook.)

Recall that a random variable X is said to have a ***Poisson distribution*** with mean $\lambda > 0$ if

$$\mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots.$$

The Poisson distribution is a common model for modeling the number of "accidents". Recall that the expected value and variance of a Poisson random variable are both equal to λ.

Example 75. Oahu has about a million cars and averages about 6 car accidents per day. The probability that any one car will have an accident is very small, and they are all (nearly) independent from each other, bus since there are so many cars, you expect 6 accidents, on average. But actual number of accidents each day is $X = 0$ or 1 or 2 or .... The number of car accidents is thus poisson distributed with mean $\lambda = 6$ What is

$$F(2) = \mathbb{P}[X \leq 2]\,?$$

The probabilities of no accidents, one accident, and two accidents are

$$p(0) = e^{-6}\frac{6^0}{0!} = e^{-6} \approx 0.002$$

$$p(1) = e^{-6}\frac{6^1}{1!} = 6e^{-6} \approx 0.015$$

$$p(2) = e^{-6}\frac{6^2}{2!} = 18e^{-6} \approx 0.045$$

Therefore $F(2) = p(0) + p(1) + p(2) \approx 0.002 + 0.015 + 0.045 = 0.062$

End of Example 75. □

In the previous lecture, we introduce the Poisson distribution as a mystery limit of coin flips, where (1) the number of coin flips is increasing and (2) the probability of heads is decreasing at a proportional rate. Let us now formalized that idea in the following theorem:

Theorem 76 (Poisson approximation)**.** *Recall that if $X \sim Binom(n, p)$, that means*

$$\mathbb{P}[X = k] = \binom{n}{k}p^k(1-p)^{n-k}, \quad \textit{for } k = 0, 1, 2, \ldots.$$

Let $\lambda = np$. When n is large and p is small (e.g., $n > 50$ and $\lambda < 5$), then we have the following approximation:

$$\mathbb{P}[X = k] \approx e^{-\lambda}\frac{\lambda^k}{k!} \quad \textit{for } k = 0, 1, 2, \ldots.$$

In other words, when n is large and p is small, X is well-approximated by a Poisson distribution with parameter $\lambda = np$.

Let's try an application of this:

Example 77 (Poisson approximation)**.** In *Dungeons and Dragons*, a 20-sided dice is used (aka the 'd20'). When you roll a 1 out of 20, that's called a ***critical failure***, which usually results in something terrible happening.

Suppose that in the course of a game, a twenty side dice is rolled $n = 100$ times. Let X be the number of critical failures. The probability of a critical failure is $p = 1/20 = .05$. So

$$X \sim \text{Bin}\left(100, \frac{1}{20}\right).$$

So, for example, we have $\mathbb{E}[X] = 100 \times \frac{1}{20} = 5$.

The pmf of X is

$$\mathbb{P}[X = k] = \binom{100}{k} \left(\frac{1}{20}\right)^k \left(1 - \frac{1}{20}\right)^{100-k}, \text{ for } k = 0, 1, 2, \ldots, 100$$

Since $p = 0.05$ is small and $n = 100$ is large, we can approximate X with a Poisson random variable Y with parameter $\lambda = np = 5$. That is, $Y \sim \text{Pois}(5)$. The pmf of Y is

$$\mathbb{P}[Y = k] = e^{-5} \cdot \frac{5^k}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

Let's compute some probabilities, rounding to three decimal places

x	$\mathbb{P}[X = k]$	$\mathbb{P}[Y = k]$
0	.006	.007
1	.031	.034
2	.081	.084
3	.140	.140
4	.178	.175

End of Example 77. □

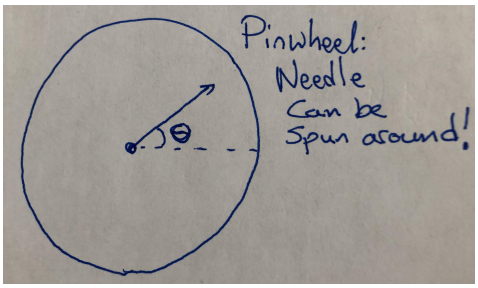
18.2 Continuous random variables

(This section is based on Chapter 4.1)

Recall that a random variable is ***discrete*** if it can assume only a finite number of values, or if its values can be listed in an infinite sequence. Chapter 4 introduces another class of random variables, which are called continuous random variables.

Definition 78 (Continuous random variable)**.** A random variable X is said to be ***continuous*** if it takes values in an interval I (or disjoint union of intervals) AND $\mathbb{P}[X = x] = 0$ for all $x \in I$.

Example 79 (Pinwheel)**.** Suppose you have a pinwheel as follows.



Suppose we spin the needle. When the needle stops spinning, we measure its angle from the dotted line. Let X be the angle of the needle. So X is a random variable. If X is measured in degrees, then

$$X \in [0, 360)$$

but A could take *any* value, not just integer values. On the other hand, if we measure X in radians, then

$$X \in [0, 2\pi).$$

Moreover we have no reason to believe any one angle is more likely than any others – for our idealized pinwheel, all values between 0 and 2π should all be equally likely.

End of Example 79. □

Definition 80 (pdf)**.** Let X be a continuous random variable. The ***probability density function*** (aka "pdf" or "density") of X is a function f_X such that for any real numbers a, b with $a \leq b$, we have

$$\mathbb{P}[X \in (a, b)] = \int_a^b f_X(x)dx$$

The graph of f_X is called the ***density curve*** of X.