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The next example presents another important discrete random variable: the geometric random variable.

17.1 Poisson random variable

Fix $\lambda \in (0,1)$. Suppose have the following coins:

- Coin #1 has probability of heads λ
- Coin #2 has probability of heads $\frac{\lambda}{2}$
- Coin #3 has probability of heads $\frac{\lambda}{3}$
- And so forth.

Consider the following sequence of experiments:

- Experiment 1: Flip coin #1 once.
- Experiment 2: Flip coin #2 twice.
- Experiment 3: Flip coin #3 three times.
- Experiment n: Flip coin #n, n times.

Let us focus now on the n^{th} experiment. Let $\xi_1, \ldots, \xi_n$ represent the n coin flips, where

$$\xi_k = \begin{cases} 1 & : \quad k^{\text{th}} \text{ coin flip is heads} \\ 0 & : \quad \text{otherwise} \end{cases}$$

for each $k \in \{1, \ldots, n\}$.

Let

$$X_n = \xi_1 + \xi_2 + \ldots + \xi_n = (\#\text{ of heads in } n \text{ flips})$$

Question: How many heads do we expect to get? We can use the linearity of expectation:

$$\begin{aligned} \mathbb{E}[X_n] &= \mathbb{E}[\xi_1 + \xi_2 + \ldots + \xi_n] \\ &= \mathbb{E}[\xi_1] + \mathbb{E}[\xi_2] + \ldots + \mathbb{E}[\xi_n] \\ &= \underbrace{\frac{\lambda}{n} + \frac{\lambda}{n} + \ldots + \frac{\lambda}{n}}_{n \text{ terms}} \\ &= \lambda. \end{aligned}$$

So the expected number of heads that we get doesn't change from experiment to experiment, even if we send $n \to \infty$. Think!—for each experiment, the probability of heads is $\frac{\lambda}{n}$ and that is getting smaller. However the total number of coin flips is getting larger and larger.

Question Is it possible that as $n \to \infty$, the random variable X_n converges to some new random variable Y, one that we haven't seen yet? What properties would that new random variable have?

- Its expectation should be λ, since $\mathbb{E}[Y] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lambda$
- It should be able to take any positive integer value, and not any other values.

Question: What do we expect the variance of Y to be?

To answer this, observe that, as the sum of n coin flips (each with $\mathbb{P}[\text{Heads}] = \frac{\lambda}{n}$), it follows by Remark 68 that $X_n \sim \text{Bin}(n, \lambda/n)$.

Applying Eq. (11) from Example 70, we find that

$$\text{Var}(X_n) = \lambda\left(1 - \frac{\lambda}{n}\right).$$

Therefore we expect the following to be true of Y:

$$\text{Var}(Y) = \lim_{n\to\infty} \text{Var}(X_n) = \lim_{n\to\infty} \lambda\left(1 - \frac{\lambda}{n}\right) = \lambda.$$

If this limit were 0, then we would expect our final result to not vary at all—in other words, we'd expect the limit to be a fixed number, rather than a random quantity. But since the limiting variance λ is positive, we expect the limit to be a random variable.

Qustion: What should the probability mass function of Y look like?

By Definition 67, the probability mass function of X_n is

$$\mathbb{P}[X_n = k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

It turns out that when n is large and p is small, the following approximation holds:

$$\binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda}\frac{\lambda^k}{k!}$$

So we might expect that in the limit, if our our coin-flipping process converges to a random variable Y, that random variable should have probability mass function

$$\mathbb{P}[Y = k] = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Theorem 71 (The exponential function)**.** *For any real number x, the* ***exponential fucntion****, denoted e^x, is the function*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Definition 72 (Poisson ("Pwason"))**.** Let $\lambda > 0$. A random variable X that takes on the values $0, 1, 2, \ldots$ is a ***Poisson*** random variable with parameter λ if its probability mass function is

$$p(k) := \mathbb{P}[X = k] = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots.$$

Before explaining this, we should check that it is actually a probability mass function: remember, it needs to sum to 1:

$$\begin{aligned}
\sum_{k=0}^{\infty} p(k) &= \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}e^{\lambda} && \text{by Theorem 71} \\
&= 1.
\end{aligned}$$

Proposition 73. *Let $X \sim Pois(\lambda)$. Then*

$$\mathbb{E}[X] = \lambda \quad \textit{and} \quad \text{Var}(X) = \lambda$$

The Poisson distribution is a common model since it models "accidents".

Example 74. Oahu has about a million cars and averages about 6 car accidents per day. The probability that any one car will have an accident is very small, and they are all (nearly) independent from each other, bus since there are so many cars, you expect 6 accidents, on average. But actual number of accidents each day is $X = 0$ or 1 or 2 or .... The number of car accidents is thus poisson distributed with mean $\lambda = 6$ What is

$$F(2) = \mathbb{P}[X \leq 2]\,?$$

The probabilities of no accidents, one accident, and two accidents are

$$p(0) = e^{-6}\frac{6^0}{0!} = e^{-6} \approx 0.002$$

$$p(1) = e^{-6}\frac{6^1}{1!} = 6e^{-6} \approx 0.015$$

$$p(2) = e^{-6}\frac{6^2}{2!} = 18e^{-6} \approx 0.045$$

Therefore $F(2) = p(0) + p(1) + p(2) \approx 0.002 + 0.015 + 0.045 = 0.062$

End of Example 74. □