16 2026-02-27 | Week 07 | Lecture 16

16.1 Application of the Law of Total Probability: The probability of winning at craps

Recall the rules of craps:

1. You roll two dice and add them up.
   - If the sum is $2, 3$, or 12, you lose ("craps")
   - If you roll 7 or 11, then you win ("natural")
   - Otherwise you establish "point", which is whatever number you got.
2. If you established point, then you must keep rolling until one of two events occurs:
   - You roll a 7: you lose
   - You roll your "point" number: you win.

Let's introduce some useful notation: Let

$$W = [\text{You win}],$$

and for each $k = 2, 3, 4, \ldots, 12$, define

$$F_k = [\text{Your first roll is } k].$$

Question: What is $\mathbb{P}[W]$?
Solution: By the *Law of Total Probability*, we have

$$\mathbb{P}[W] = \sum_{k=2}^{12} \mathbb{P}[W \mid F_k] \times \mathbb{P}[F_k] \tag{9}$$

In words,

$$\mathbb{P}[W \mid F_k] = \text{ probability of winning, given you rolled } n \text{ on the first roll}$$

k	$P[W \mid F_i]$	$\mathbb{P}[F_k]$
2	0	1/36
3	0	2/36
4	1/3	3/36
5	4/10	4/36
6	5/11	5/36
7	1	6/36
8	5/11	5/36
9	4/10	4/36
10	1/3	3/36
11	1	2/36
12	0	1/36

How do we get, say, $\mathbb{P}[W \mid F_4]$?
(Method 1: The long way) Let's enumerate the ways we can win:

4	win on the first roll after establishing point
$*4$	win on second roll
$**4$	win on third roll
$***4$	etc
$****4$	
$\ldots$	

where $*$ denotes any roll other than a 4 or a 7.

These have probabitilies

$$\begin{aligned}
\mathbb{P}[4] &= \frac{3}{36} \\
\mathbb{P}[*4] &= \frac{27}{36}\frac{3}{36} \\
\mathbb{P}[**4] &= \left(\frac{27}{36}\right)^2 \frac{3}{36} \\
\mathbb{P}[***4] &= \left(\frac{27}{36}\right)^3 \frac{3}{36} \\
\\
\mathbb{P}[****4] &= \left(\frac{27}{36}\right)^4 \frac{3}{36} \\
\\
&\dots
\end{aligned}$$

So

$$\begin{aligned}
\mathbb{P}[W \mid F_4] &= \frac{3}{36} + \frac{27}{36}\frac{3}{36} + \left(\frac{27}{36}\right)^2 \frac{3}{36} \\
&= \frac{3}{36}\left(1 + \frac{27}{36} + \left(\frac{27}{36}\right)^2 + \dots\right) \\
&= \frac{3}{36} \cdot \frac{1}{1 - \frac{27}{36}} \\
&= \frac{3}{9} \\
&= \frac{1}{3}.
\end{aligned}$$

(Method 2: The short way). Given F_4, you know you're gonna keep rolling until you get a 4 or a 7, which is eventually going to happen. And whether you win or lose is determined by what you roll on that last roll. Your probability of winning is the probability of rolling a 4 on your last roll, given that your last roll is a 4 or a 7. Thus,

$$\begin{aligned}
\mathbb{P}[W \mid F_4] &= \mathbb{P}[\text{roll 4} \mid \text{roll 4 or 7}] \\
&= \frac{\mathbb{P}[(\text{roll 4}) \text{ and } (\text{roll 4 or 7})]}{\mathbb{P}[\text{roll 4 or 7}]} && \text{by def of conditional prob.} \\
&= \frac{\mathbb{P}[\text{roll 4}]}{\mathbb{P}[\text{roll 4 or 7}]} \\
&= \frac{3/36}{3/36 + 6/36} \\
&= \frac{3}{9} && = \frac{1}{3}.
\end{aligned}$$

Continue in this manner to get all the values of the table. Then plugging the values from the table into Eq. (9) gives

$$\mathbb{P}[W] = 0 \cdot \frac{1}{36} + 0 \cdot \frac{2}{36} + \frac{1}{3} \cdot \frac{3}{36} + \frac{4}{10} \cdot \frac{4}{36} + \frac{5}{11} \cdot \frac{5}{36} + 1 \cdot \frac{6}{36} + \ldots + 0 \cdot \frac{1}{36} = 0.492999...$$

Your chance of winning is about 49.3%.

16.2 Binomial random variable (continued)

(*Based on chapter 3.4 in the textbook.*)

Recall that a random variable is called a ***binomial random variable*** with number of trials n and success probability $p \in (0,1)$ if

$$\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for every $k \in \{0, \ldots, n\}$.

Interpretation: If p is the probability of a coin landing on heads, then

$$X = (\#\text{ of heads in } n \text{ coin tosses}).$$

Example 70 (Variance of a binomial random variable)**.** Let X be a binomial random variable. In the last lecture we showed that

$$\mathbb{E}[X] = np.$$

Question: What is the variance of X?

Solution: Recall that we can write

$$X = \xi_1 + \ldots + \xi_n,$$

where

$$\xi_i = \begin{cases} 1 & : \text{ with probability } p \\ 0 & : \text{ with probability } 1-p \end{cases}.$$

Therefore, using the additivity of variance (Theorem 66), we have

$$\mathrm{Var}(X) = \mathrm{Var}(\xi_1 + \ldots + \xi_n) = \mathrm{Var}(\xi_1) + \ldots + \mathrm{Var}(\xi_n). \tag{10}$$

We need to compute $\mathrm{Var}(\xi_i)$, for $i = 1, \ldots, n$. Observe that by the shortcut formula for variance (Theorem 64, for every i, we have

$$\begin{aligned} \mathrm{Var}(\xi_i) &= \mathbb{E}\left[\xi_i^2\right] - \left(\mathbb{E}[\xi_i]\right)^2 \\ &= p - p^2 \\ &= p(1-p). \end{aligned}$$

Plugging this into Eq. (10) implies that

$$\mathrm{Var}(X) = np(1-p).$$

End of Example 70. □