9 2026-02-02 | Week 04 | Lecture 09

9.1 Conditional probability

This section is based on section 2.4 in the text.

Definition 29 (Conditional probability)**.** Let A, B be events, and assume that $\mathbb{P}[A] > 0$. Then the ***conditional probability of B, given A***, denoted $\mathbb{P}[B \mid A]$, is given by the formula

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}.$$

Interpretation: $\mathbb{P}[A \mid B]$ is the probability of A when we know that event B happened.

Multiplication rule: $\mathbb{P}[A \cap B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$

Definition 30. We say that there exists a ***positive relationship*** between events A and B if

$$\mathbb{P}[B \mid A] > \mathbb{P}[B],$$

and a ***negative relationship*** if

$$\mathbb{P}[B \mid A] < \mathbb{P}[B].$$

- Smoking and getting lung canger have a positive relationship: if you know someone is a smoker (A), then the probability that they get lung cancer (B) is higher than if you didn't know they were a smoker.

Example 31 (Conditional probability)**.** I roll a dice behind a screen. I tell you that I rolled an even number. What's the probability that the dice roll is a 4 or a 6?

Solution: We have $S = \{1,2,3,4,5,6\}$, $A = \{2,4,6\}$, $B = \{4,6\}$. We want to compute $\mathbb{P}[B \mid A]$. By Definition 29,

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[AB]}{\mathbb{P}[A]} = \frac{2/6}{1/2} = \frac{2}{3}.$$

Put differently, we have three outcomes in A, which are all equally likely. Two of them (4 and 6) mean that event B occurs. So by the enumeration principle, the probability is 2/3.

End of Example 31. □

Example 32 (Similar to examples 2.27 and 2.28 in the textbook)**.** A vampire goes to the blood bank looking to find some type O+ blood (the most delicious type). He finds 4 unlabeled bags of blood. Only one of the bags is O+, but the vampire doesn't know which one, so he resorts to taste-testing to find the O+ bag.

(a) What is the probability that he must test at least 3 bags to find the desired type?

Solution: Let X be the number of tested bags. We want to find $\mathbb{P}[X \geq 3]$. Let $A =$ [first bag isn't O+]. Let $B =$ [second bag isn't O+]

$$\begin{aligned}\mathbb{P}[X \geq 3] &= \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A]\,\mathbb{P}[B \mid A] \\ &= \frac{3}{4} \cdot \frac{2}{3} \\ &= \frac{1}{2}.\end{aligned}$$

(b) What's the probability that he must test exactly 3 bags?

Solution: We want $\mathbb{P}[X = 3] = \mathbb{P}[$first bag isn't O+ $\cap$ second bag isn't O+ $\cap$ third bag is O+$]$.

$$\begin{aligned}\mathbb{P}[X=3] &= \mathbb{P}[\text{third is} \mid \text{first isn't} \cap \text{second isn't}] \times \mathbb{P}[\text{second isn't} \mid \text{first isn't}] \cdot \mathbb{P}[\text{first isn't}] \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \\ &= \frac{1}{4}\end{aligned}$$

Here we have used the following multiplication rule:

$$\begin{aligned}\mathbb{P}[A \cap B \cap C] &= \mathbb{P}[C \mid A \cap B]\,\mathbb{P}[A \cap B] \\ &= \mathbb{P}[C \mid A \cap B]\,\mathbb{P}[B \mid A]\,\mathbb{P}[A]\end{aligned}$$

with $A =$ [first bag isn't O+], $B =$ [second bag isn't O+], and $C =$ [third bag is O+].

End of Example 32. □