8 2025-01-30 | Week 03 | Lecture 08

This lecture is based on sections 2.3 and 2.4 in the textbook.

8.1 Applications of the 3 combinatorial principles

Example 26. Let $S = \{1, 2, 3, 4, 5, 6\}$. How many unordered subsets (i.e., combinations) of size 2 are there?

$$\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)} = \frac{6 \cdot 5}{2} = 15.$$

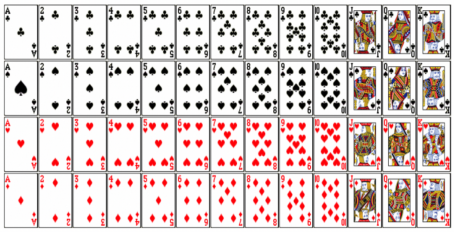
The fiften combinations are

$$\begin{aligned} \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\} \\ \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\} \\ \{3,4\}, \{3,5\}, \{3,6\} \\ \{4,5\}, \{4,6\} \\ \{5,6\} \end{aligned}$$

note that the order doesn't matter, i.e., $\{2,1\} = \{1,2\}$ so this just counts as one, not two.

End of Example 26. □

Example 27 (Poker)**.** A poker hand consist of 5 randomly chosen cards from a standard 52-card deck.



Some questions about poker hands:

1. How many distinct poker hands are there?

$$\binom{52}{5} = \frac{52!}{5! \cdot 47!} = 2,598,960$$

2. A **flush** occurs when all five cards are the same suit. How many ways are there to get a flush?

$$4 \times \binom{13}{5} = 4 \cdot \frac{13!}{5!\,8!} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 5148$$

   The 4 is to choose a suit. Then we choose 5 out of 13 cards in that suit.

3. What's the probability of getting a flush?

   To answer this, we use the enumeration principle:

$$\frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}} = \frac{5,148}{2,598,960} \approx 0.002$$

   In other words, the probability of a flush is pretty low: about 0.2%.

End of Example 27. □

8.2 Introduction to craps

Craps is a dice-rolling game in which a player, known as the shooter, rolls a pair of 6-sided dice. The shooter wins, loses or re-rolls depending on what happens.

Here's a set of rules for the shooter:

1. You roll two dice and add them up.
   - If the sum is $2, 3$, or 12, you lose ("craps")
   - If you roll 7 or 11, then you win ("natural")
   - Otherwise you establish "point", which is whatever number you got.
2. If you established point, then you must keep rolling until one of two events occurs:
   - You roll a 7: you lose
   - You roll your "point" number: you win.

Tallying the results, we saw that shooters won about 50% of the time. Is this a fair game? We will set out to answer this question. To do so, we'll need to introduce some new ideas. A key idea is that of conditional probability:

8.3 Conditional probability

This section is based on section 2.4 in the text.

Definition 28 (Conditional Probability)**.** Let A, B be events, and assume that $\mathbb{P}[A] > 0$. Then the ***conditional probability of B, given A***, denoted $\mathbb{P}[B \mid A]$, is given by the formula

$$\mathbb{P}[B \mid A] := \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]}.$$