Math 372: Homework 07

Due Thursday, March 12

Instructions: Do not write your solutions on this sheet—you don't have enough space to show your work. So use separate sheets of paper. For all problems you need to show your work.

Problem 1. The continuous random variable X has probability density function

$$f(x) = \begin{cases} 1/5 & : \quad 0 < x < 1 \\ 2/5 & : \quad 1 < x < 3 \\ 0 & : \quad \text{elsewhere} \end{cases}$$

Let n be a positive integer. Find $\mathbb{E}[X^n]$.

Problem 2. Let $f(x) = 3x^2$ for all x with $0 < x < 1$.

(a) Show that f is a valid pdf.

(b) Suppose Y is a random variable whose pdf is f. Find $\mathbb{E}[Y]$.

(c) Find the variance of Y.

Note: In the next two problems, we'll use the indicator variable notation $\mathbf{1}_{[t\geq 0]}$, which means the following:

$$\mathbf{1}_{[t\geq 0]} = \begin{cases} 1 & : \quad t \geq 0 \\ 0 & : \quad t < 0 \end{cases}$$

Problem 3 (Waiting times 1/2)**.** Suppose you station yourself at a spot on an idyllic country road and watch as cars pass by. Let T be the waiting time, in minutes, between the first and second cars to pass by. Assume T is an exponentially distributed random variable with rate $\lambda > 0$. In other words, the pdf of T is

$$f_T(t) = \lambda e^{-\lambda t}\mathbf{1}_{[t\geq 0]} = \begin{cases} \lambda e^{-\lambda t} & : \quad t \geq 0 \\ 0 & : \quad t < 0 \end{cases}$$

(a) Compute $\mathbb{E}[T]$ using the formula $\mathbb{E}[T] = \int_{-\infty}^{\infty} t f_T(t) dt$.

(b) Suppose that $\lambda = 3$ cars per minute. What is the average waiting time, in seconds, for a car to pass by?

(c) Show that $\mathbb{P}[T > s + t \mid T > s] = \mathbb{P}[T > t]$ for all $t, s > 0$.

(d) Interpret the equation from Part (c) in words. *Hint: compare with proble 2c from homework 6.*

Problem 4 (Waiting times 2/2)**.** Let T be the time, in minutes, between successive drink orders at a bar. Suppose that a reasonable probability model for T is given by the pdf

$$f_T(t) = Ce^{-\frac{t}{4}}\mathbf{1}_{[t\geq 0]},$$

where is $C > 0$ is some number that you'll have to determine.

(a) Find a value for C such that $f_T(t)$ is a valid pdf. (Hint: use the fact that in order to be a valid pdf, the function f_T must satisfy $\int_{-\infty}^{\infty} f_T(t)dt = 1$).

(b) What is the probability that the time between successive orders is greater than 4 minutes?

(c) On average, how long must the bartender wait between drink orders?

Problem 5. Let X denote the amount of time a book on two-hour reserve is actually checked out, and supposed the cdf of X is

$$F(x) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x^2}{4} & : \quad 0 \leq x < 2 \\ 1 & : \quad 2 \leq x \end{cases}$$

(a) Find $\mathbb{P}[X \leq 1]$

(b) Find $\mathbb{P}[.5 \leq X \leq 1]$

(c) What is the median checkout duration of $\tilde{\mu}$? [In other words, solve $F(\tilde{\mu}) = .5$.]

(d) Obtain the density function $f(x)$

(e) Calculate $\mathbb{E}[X]$

(f) Calculate $\mathrm{Var}(X)$.

(g) If the borrower is charged an an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $\mathbb{E}[h(X)]$.

Problem 6. Let T be the waiting time until the lightbulb spontaneoulsy burns out. Assume that T is exponentially distributed and that the expected lifetime of the bulb is 2 years.

(a) What is the rate parameter of T?

(b) What is the probability density function of T? What is the cumulative density function?

(c) Compute $\mathbb{P}[1 < T < 2]$.

(d) Let σ denote the standard deviation of T. Compute σ.

(e) Compute $\mathbb{P}[T \leq 4\sigma]$

(f) Compute $\mathbb{P}[T > 2]$.

(g) Compute $\mathbb{P}[T > 4 \mid T > 2]$.

(h) Using several complete sentences, interpret your answers to parts (f) and (g). Is there a contradiction?

Problem 7 (Exercise 4.1.2)**.** Suppose the reaction temperature X (in celsius) in a certain chemical process has uniform distribution with $A = -5$ and $B = 5$.

(a) Compute $\mathbb{P}[X < 0]$

(b) Compute $\mathbb{P}[-2.5 < X < 2.5]$

(c) Compute $\mathbb{P}[-2 \leq X \leq 3]$

(d) For k satisfying $-5 < k < k + 4 < 5$, compute $\mathbb{P}[k < X < k + 4]$.

Problem 8. The lifetime of a gadget costing \$300 is exponentially distributed with mean 3 years. The manufacturer agrees to pay a full refund to a buyer if the gadget fails during the first year following the purchase, a two-thirds refund if it fails between the first and second year, and a one-third refund if it fails between the second and third year. If the manufacturer sells 1000 gadgets, how much should it expect to pay in refunds?

Problem 9 (Extra credit)**.** An 'exploding' dice roll is a dice mechanic in various boad games whereby a player gets to roll a dice, and then if they rolled the maximum number on the die, they get to roll it again and add the new roll to the old one. This can happen more than once. Compute the expectation of an exploding 6-sided dice.