Math 372: Homework 10

Due Wednesday, November 4

Calculations with pdfs

Problem 1. State precise mathematical definitions of the following terms: pmf, pdf, and cdf.

Problem 2 (continuous r.v.)**.** The continuous random variable X has probability density function

$$f(x) = \begin{cases} 1/3 & : \quad 0 < x < 1 \\ 1/6 & : \quad 1 < x < 5 \\ 0 & : \quad \text{elsewhere} \end{cases}$$

(a) Compute $\mathbb{E}[X]$ and $\mathbb{E}\left[X^2\right]$

(b) Compute $\text{Var}(X)$

(c) Compute $\mathbb{E}\left[(X+1)^2\right]$

Problem 3. The Pareto distribution is a "power-law" distribution used for modeling things like income distribution and earthquake magnitudes. Specifically, given a random variable X, we say that the distribution of X is a member of the *Pareto family of distributions* if its cdf is

$$F(x) = \mathbb{P}[X \leq x] = \begin{cases} 1 - \left(\frac{b}{x}\right)^a & : \quad x \geq b \\ 0 & : \quad x < b \end{cases}$$

for some fixed positive parameters a and b.

(a) Derive the pdf of X. Make sure to specify the function on all parts of its domain: both when $x \geq b$ and when $x < b$.

(b) Show that the improper integral $\int_1^\infty \frac{1}{x^c} dx$ diverges to $+\infty$ when $c \leq 1$, but converges to $\frac{1}{c-1}$ when $c > 1$. *Hint:* recall that

$$\int_1^\infty \frac{1}{x^c} dx = \lim_{y \to \infty} \int_1^y \frac{1}{x^c} dx.$$

(c) What is $\mathbb{E}[X]$? Consider the cases with $a \leq 1$ and $a > 1$ separately.

Problem 4 (Exercise 4.4.61)**.** Data collected at Toronto Pearson International Airport suggests that an exponential distribution is a good model for rainfall duration, and that the mean rainfall duration is 2.725 hours.

(a) What is the probability that the duration of a particular rainfall event at this location is at least 2 hours? At most 3 hours? Between 2 and 3 hours?

(b) What is the probability that the rainfall duration exceeds the mean value by more than 2 standard deviations?

Problem 5 (Exercise 4.1.4)**.** Let X denote the vibratory stress (measured in `psi`) on a wind turbine blade at a particular wind speed in a wind tunnel. The article "Blade Fatigue Life Assessment with Application to VAWTS" *(J. of Solar Energy Engr., 1982)* proposes the *Rayleigh distribution*, with pdf

$$f(x) = \begin{cases} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & : \quad x > 0 \\ 0 & : \quad x \leq 0 \end{cases}$$

as a model for the X distribution. (Here, $\theta > 0$ is a parameter.)

(a) Verify that $f(x)$ is a legitimate pdf for any $\theta > 0$.

(b) Suppose $\theta = 100$ (a value suggested in the article). What is the probability that X is at most 200? Less than 200? At least 200?

(c) What is the probability that X is between 100 and 200 (again assuming $\theta = 100$)?

(d) Give an expression for $\mathbb{P}[X \leq x]$.

Joint distributions

Problem 6. Let (X, Y) be uniformly distributed on the unit square $0 < x < 1$, $0 < y < 1$. Find the following probabilities: *[hint: sketch the regions in the unit square]*

(a) $\mathbb{P}[X < Y]$

(b) $\mathbb{P}[X + Y < 1]$

(c) $\mathbb{P}\left[\min(X, Y) > \frac{3}{4}\right]$

(d) $\mathbb{P}\left[XY \leq \frac{1}{2}\right]$

Problem 7 (Exercise 5.1.3)**.** A grocery store has both an express checkout lane and a superexpress checkout lane. Let X_1 denote the number of customers in line at the express checkout at a particular time of day, and let X_2 denote the number of customers in line at the superexpress checkout at the same time. Suppose the joint pmf of X_1 and X_2 is given by the following table:

	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$
$x_1 = 0$	0.08	0.06	0.05	0.00	0.00
$x_1 = 1$	0.07	0.15	0.04	0.03	0.01
$x_1 = 2$	0.04	0.05	0.10	0.04	0.05
$x_1 = 3$	0.00	0.04	0.06	0.07	0.06

(a) Compute $\mathbb{P}[X_1 = 1, X_2 = 1]$, the probability that there is exactly one customer in each line?

(b) What is $\mathbb{P}[X_1 = X_2]$, that is, the probability that the number of customers in the two lines is identical?

(c) Let A denote the event that there are at least two more customers in one line than in the other. Express A in terms of X_1 and X_2 and calculate the probability of this event.

(d) What is the probability that the total number of customers in the two lines is exactly four? At least four?

Problem 8 (Exercise 5.1.4)**.** Return to the situation described in Problem 7.

(a) Determine the marginal pmf of X_2, and then calculate the expected number of customers in line at the superexpress checkout.

(b) Determine the marginal pmf of X_1.

(c) By inspection of the probabilities $\mathbb{P}[X_2 = 4]$, $\mathbb{P}[X_1 = 0]$, and $\mathbb{P}[X_2 = 4, X_1 = 0]$, are X_1 and X_2 independent random variables? Explain.

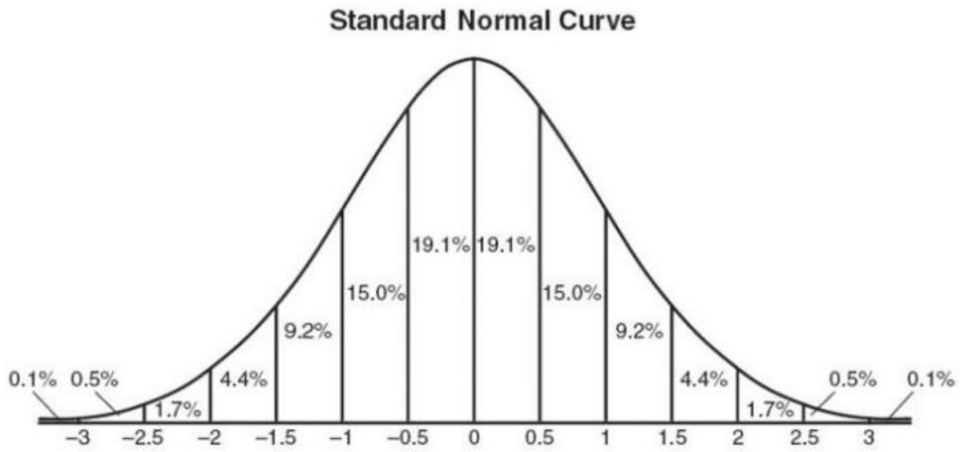
Problem 9. Annie and Alvie have agreed to meet for lunch between noon and 1:00pm. Denote Annie's arrival time by X and Alvie's by Y, where arrival time is measured in *hours after noon.* Suppose that X and Y are independent with pdfs

$$f_X(x) = \begin{cases} 3x^2 & : \ 0 \leq x \leq 1 \\ 0 & : \ \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : \ 0 \leq y \leq 1 \\ 0 & : \ \text{otherwise} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person? *Hint:* $h(X, Y) = |X - Y|$

Normal distribution

For the remaining problems, the following curve of the density of a standard normal random variable Z may be helpful:



Problem 10 (Normal distribution)**.** Resting breathing rates for college-age students are approximately normally distributed with mean 12 and standard deviation 2.3 breaths per minute. What fraction of all college-age students have breathing rates in the following intervals?

(a) 9.7 to 14.3 breaths per minute

(b) 7.4 to 16.6 breaths per minute

(c) 9.7 to 16.6 breaths per minute

(d) Less than 5.1 or more than 18.9 breaths per minute

Problem 11 (Exercise 4.3.41)**.** The automatic opening device of a military cargo parachute has been designed to open when the parachute is 200 m above the ground. Suppose opening altitude actually has a normal distribution with mean value 200 m and standard deviation 30 m. Equipment damage will occur if the parachute opens at an altitude of less than 100 m. What is the probability that there is equipment damage to the payload of at least one of five independently dropped parachutes?

Problem 12 (Exercise 4.3.46)**.** The Rockwell hardness of a metal is determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose the Rockwell hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

(a) If a specimen is acceptable only if its hardness is between 67 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

(b) If the acceptable range of hardness is $(70 - c, 70 + c)$, for what value of c would 95% of all specimens have acceptable hardness?

(c) If the acceptable range is as in part (a) and the hardness of each of ten randomly selected specimens is independently determined, what is the expected number of acceptable specimens among the ten?

(d) What is the probability that at most eight of ten independently selected specimens have a hardness of less than 73.84? [Hint: If Y is the number among the ten specimens with hardness less than 73.84, then Y is a binomial variable; what is p?]