Math 372: Homework 09

Due Monday, October 26

Problem 1. Let T be the time, in minutes, between successive drink orders at a bar. Suppose that a reasonable probability model for T is given by the pdf

$$f_T(t) = \begin{cases} Ce^{-\frac{t}{4}} & : \quad t \geq 0 \\ 0 & : \quad t < 0, \end{cases}$$

where is $C > 0$ is some number that you'll have to determine.

(a) Find a value for C such that $f_T(t)$ is a valid pdf.

(b) What is the probability that the time between successive orders is greater than 4 minutes?

(c) On average, how long must the bartender wait between drink orders?

Problem 2. Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf of X is

$$F(x) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x^2}{4} & : \quad 0 \leq x < 2 \\ 1 & : \quad 2 \leq x \end{cases}$$

(a) Find $\mathbb{P}[X \leq 1]$

(b) Find $\mathbb{P}[.5 \leq X \leq 1]$

(c) Let m be the median checkout duration (i.e., m is the number such that $F(m) = .5$). Find m.

(d) Find the density function $f(x)$

(e) Calculate $\mathbb{E}[X]$

(f) Calculate $\mathrm{Var}(X)$.

(g) If the borrower is charged an an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $\mathbb{E}[h(X)]$.

Problem 3. Let T be the waiting time until a lightbulb spontaneoulsy burns out. Assume that T is exponentially distributed and that the expected lifetime of the bulb is 2 years.

(a) What is the rate parameter λ of T?

(b) What is the probability density function of T? What is the cumulative density function?

(c) Compute $\mathbb{P}[1 < T < 2]$.

(d) Let σ denote the standard deviation of T. Compute σ.

(e) Compute $\mathbb{P}[T \leq 4\sigma]$

(f) Compute $\mathbb{P}[T > 2]$.

(g) Compute $\mathbb{P}[T > 4 \mid T > 2]$.

(h) Using several complete sentences, interpret your answers to parts (f) and (g). Is there a contradiction?

Problem 4. Let X be a random variable with Poisson distribution with parameter $\lambda > 0$. Using the formula $\mathbb{E}[X] = \sum_{k=0} k\mathbb{P}[X = k]$ show that $\mathbb{E}[X] = \lambda$. [Hint: use the formula $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.]

Problem 5. Suppose X is a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & : \quad 0 < x < 1 \\ 0 & : \quad \text{otherwise} \end{cases}$$

(a) Find the cumulative density function $F(x)$. *(Make sure to consider both positive and negative x.)*

(b) What is $\mathbb{P}[0.1 < X < 0.5]$?

(c) What is $\text{Var}(X)$?

(d) What is $\mathbb{E}\left[(X+3)^2\right]$?

Problem 6 (Exercise 4.1.2)**.** Suppose the reaction temperature X (in celsius) in a certain chemical process has uniform distribution with $A = -5$ and $B = 5$.

(a) Compute $\mathbb{P}[X < 0]$

(b) Compute $\mathbb{P}[-2.5 < X < 2.5]$

(c) Compute $\mathbb{P}[-2 \leq X \leq 3]$

(d) For k satisfying $-5 < k < k+4 < 5$, compute $\mathbb{P}[k < X < k+4]$.

Problem 7. The lifetime of a solid-state hard drive costing \$300 is exponentially distributed with mean 3 years. The manufacturer agrees to pay a full refund to a buyer if the hard drive fails during the first year following the purchase, a two-thirds refund if it fails between the first and second year, and a one-third refund if it fails between the second and third year. If the manufacturer sells 1000 hard drives, how much should it expect to pay in refunds?

Problem 8 (Exercise 4.3.32)**.** Suppose the force acting on a column that supports a building is normally distributed with mean value 15 kips and standard deviation 1.25 kips. Compute the following probabilities by standardizing and then inspecting the standard normal curve shown below:

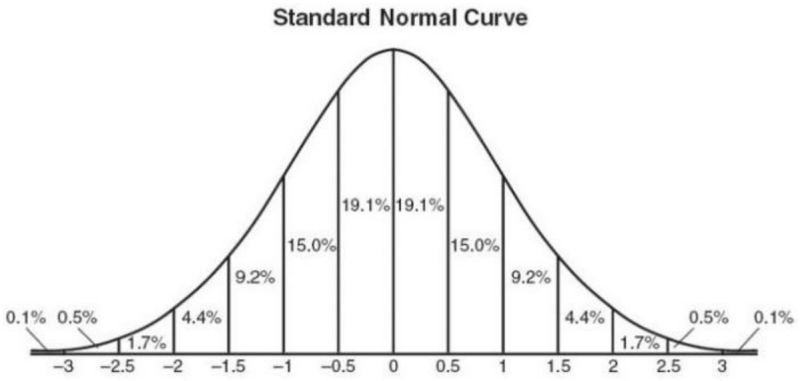
(a) $\mathbb{P}[X \leq 15]$

(b) $\mathbb{P}[X \leq 17.5]$

(c) $\mathbb{P}[X \geq 10]$

(d) $\mathbb{P}[14 \leq X \leq 18]$

(e) $\mathbb{P}[|X - 15| \leq 3]$



Problem 9. The density function for the standard normal distribution is the function

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

By computing first and second derivatives, show that ϕ is concave down for $|z| < 1$, concave up for $|z| > 1$, with inflection points at $z = \pm 1$. (Recall: a function f is concave up at x if $f''(x) > 0$, and concave down if $f''(x) < 0$.)