Math 372: Homework 08

Due Wednesday, October 21

Problem 1. A bank offers two service counters: (1) a drive-up counter, and (2) a walk-in counter. In a random workday, let X be the proportion of time that the drive-up counter is in use, and Y the proportion of time that the walk-in counter is in use. Assume that the joint pdf of (X, Y) is

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & : \quad 0 \le x, y \le 1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

(a) Verify that f is a valid pdf.

(b) Find the probability that neither counter is in use more than one-quarter of the time.

(c) Find the marginal densities f_X and f_Y, and determine whether X and Y are independent.

(d) Using f_Y, compute $\mathbb{P}[.25 \le Y \le .75]$.

Problem 2 (Waiting times)**.** Suppose you station yourself at a spot on an idyllic country road and watch as cars pass by. Let T be the waiting time, in minutes, between the first and second cars to pass by. Assume $T \sim \exp(\lambda)$, i.e., that the pdf of T is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & : \quad t \geq 0 \\ 0 & : \quad t < 0. \end{cases}$$

(a) Suppose the rate of cars passing by is $\lambda = 3$ cars per minute. What is the average waiting time, in seconds, for a car to pass by?

(b) Show that $\mathbb{P}[T > s + t \mid T > s] = \mathbb{P}[T > t]$ for all $t, s > 0$.

(c) Interpret the equation from Part (b) in words. *(Hint: compare with proble 8d from homework 6.)*

Problem 3 (Exercise 5.1.1)**.** A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y, denoted $p(x, y)$ is described by the following table:

		Y		
		0	1	2
X	0	0.10	0.08	0.06
	1	0.04	0.20	0.14
	2	0.02	0.06	0.30

(a) What is $\mathbb{P}[X = 1 \text{ and } Y = 1]$?

(b) Compute $\mathbb{P}[X \leq 1 \text{ and } Y \leq 1]$.

(c) Using words, describe the event $[X \neq 0 \text{ and } Y \neq 0]$, and then compute its probability.

(d) Compute the marginal pmfs of $p_X(x)$ and $p_Y(y)$ by calculating row and column sums of the table. Write the sums in the margins of the above table.

(e) Using $p_X(x)$, what is $\mathbb{P}[X \leq 1]$?

(f) Are X and Y independent random variables? Justify your answer.

Problem 4. You have two lightbulbs. Let X be the lifetime of the first, and Y the lifetime of the second (assume X, Y are measured in years). Suppose X and Y are independent and that $X, Y \sim \exp(1)$.

(a) What is the joint pdf of (X, Y)?

(b) What is the probability that both bulbs burn out within the first year?

(c) What is the probability that the total lifetime of the two bulbs is at most 2 years? (*Hint: draw a picture of the region* $A = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 2\}$.)

(d) What is the probability that the total lifetime is between 1 and 2 years?

Problem 5. Wingsuit gliding accidents at a given jumpsite are classified as either 'fatal' or 'nonfatal'. Denote the number of fatal accidents over the course of a year by X, and the number of nonfatal accidents by Y. Suppose the joint pmf of (X, Y), denoted $p(x, y)$, is is given by the following table:

		Y					
		0	1	2	3	4	5
	0	.025	.050	.125	.150	.100	.050
X	1	.015	.030	.075	.090	.060	.030
	2	.010	.020	.050	.060	.040	.020

(a) Find the marginal distributions p_X and p_Y. Write them in the margins of the table above.

(b) What is the probability that there are exactly three accidents? At least 3 accidents?

(c) What is the probability of at least one fatal accident?

(d) What is the probability that fatal accidents outnumber nonfatal accidents?

(e) Are X and Y independent?

Problem 6. The continuous random variable X has probability density function

$$f(x) = \begin{cases} 1/5 & : \quad 0 < x < 1 \\ 2/5 & : \quad 1 < x < 3 \\ 0 & : \quad \text{elsewhere} \end{cases}$$

Let n be a positive integer. Find $\mathbb{E}[X^n]$.