HW 6 Solutions

Problem 1 Let $\mu = E[X]$. Then:

$$\begin{aligned}
Var(X) &= E[(X-\mu)^2] \\
&= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - 2\mu \underbrace{E[X]}_{=\mu} + \mu^2 \qquad \text{(Linearity of expectation)} \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2 \\
&= E[X^2] - (E[X])^2, \quad \text{since } \mu = E[X].
\end{aligned}$$

Problem 2

(a) $E[K] = 0(.51) + 1(.33) + 2(.11) + 3(.04) + 4(.01)$

$= .71$

(b) $\text{Var}(K) = E[K^2] - (.71)^2$, so we need to compute $E[K^2]$.

$E[K^2] = 1(.33) + 4(.11) + 9(.04) + 16(.01)$

≈ 1.29

So $\text{Var}(K) = 1.29 - (.71)^2$

$= .7859.$

(c) $\lambda = .7$, so $q(k) = \frac{e^{-.7}(.7)^k}{k!}$, $k \in \{0, 1, 2, \ldots\}$

k	p(k)	q(k)
0	.51	.497
1	.33	.348
2	.11	.12
3	.04	.028
4	.01	.005
≥ 5	0	.002

The probabilities are very close!

Problem 3 Let X be the dice roll, so

$$X \in \{1, 2, 3, 4, 5, 6\}.$$

Let $A = [X \leq 3]$

$B = [X \text{ is even}]$

Then $P[B] = \frac{1}{2}$.

But $P[B|A] + P[B|A^c] = \frac{1}{3} + \frac{2}{3} = 1$.

Problem 4

(a) $E[X] = 3.5$

$E[X^2] = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

$Var(X) = \frac{91}{6} - (3.5)^2 \approx 2.91667$

(b) $E[Y] = \frac{1}{2}$

$E[Y^2] = \frac{1}{2}$

$Var(Y) = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$

(c.) $E[Z] = P$

$E[Z^2] = P$

$Var(Z) = P - P^2 = P(1-P)$.

Problem 5

$P[\text{Alice loses}] = P[\text{all 6 dice are not '6'}] = \left(\frac{5}{6}\right)^6$

So $P[\text{Alice wins}] = 1 - \left(\frac{5}{6}\right)^6 \approx .665$

Bob loses if he rolls 0 or 1 sixes. (disjoint!)

$\therefore P[\text{Bob lose}] = P[\text{Bob rolls 0 sixes}] + P[\text{Bob rolls 1 six}]$

$$= \left(\frac{5}{6}\right)^{12} + \binom{12}{1}\frac{1}{6}\left(\frac{5}{6}\right)^{11}$$

$$= \left(\frac{5}{6}\right)^{12} + 2\left(\frac{5}{6}\right)^{11}$$

$$\approx .3813.$$

So $P[\text{Bob Wins}] \approx 1 - .3813 = .6187$

So Alice is slightly more likely to win.

Problem 6

(a) $E[X] = 16(.2) + 18(.5) + 20(.3) = 18.2$ cubic feet.

$E[X^2] = 16^2(.2) + 18^2(.5) + 20^2(.3) = 333.2$

$\text{Var}(X) = E[X^2] - (E[X])^2$

$= 333.2 - (18.2)^2$

$= 1.96$.

~~(b) Let $h(x) = 70x - 650$. We want to compute~~

(b) We want to compute

$E[70X - 650]$.

By linearity of expectation,

$E[70X - 650] = 70\,E[X] - 650$

$= 70(18.2) - 650$ by part (a)

$= 624$ dollars.

(c) $\text{Var}(70X - 650) = \text{Var}(70X)$

$= 70^2\,\text{Var}(X)$ ← using properties of variance from lecture

$= 70^2(1.96)$ by part (a)

$= 9604$

(d) $E[h(X)] = E[X] - .008\,E[X^2] = 18.2 - .008(333.2) = 15.5344$.

↑ by Part (a)

Problem 7 Let X be the final score of the exploding dice.

$$\therefore E[X] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\underbrace{(6 + E[X])}_{\text{if you roll a 6}}.$$

Simplify:

$$E[X] = \frac{21}{6} + \frac{1}{6}E[X]$$

$$\Rightarrow \quad \frac{5}{6}E[X] = \frac{21}{6}$$

$$\Rightarrow \quad E[X] = \frac{21}{5} \approx 4.2.$$

Problem 8

(a) Assuming the shots are independent,

$$P[X=k] = \underbrace{(1-p)^{k-1}}_{\text{miss the first } k-1 \text{ shots}} \underbrace{p}_{\text{make the } k\text{th shot}}$$

(b) The event $[X>k]$ occurs precisely when the shooter misses the first k shots. Hence,

$$P[X>k] = (1-p)^k, \qquad k \in \{0,1,2,\ldots\}.$$

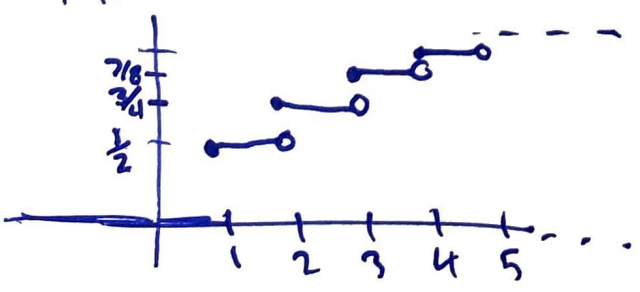
Hence $P[X \le k] = 1-(1-p)^k$.

~~(c) $F(x) = P[X \le x] = \begin{cases} 0 & \text{if } x<1 \\ 1-(1-p)^k & \text{if } k \le x < k+1, \text{ for } k \in \{1,2,\ldots\} \end{cases}$~~

(c) Here, with $p = \frac{1}{2}$,

$$F(x) = P[X \le x] = \begin{cases} 0 & \text{if } x<1 \\ \frac{1}{2} & \text{if } 1 \le x < 2 \\ 1-\frac{1}{4} & \text{if } 2 \le x < 3 \\ 1-\frac{1}{8} & \text{if } 3 \le x < 4 \\ 1-\frac{1}{16} & \text{if } 4 \le x < 5 \\ \vdots & \end{cases}$$

Plot



Problem 8

(d) This is the most interesting part! ☺

Let $m > 0$ and $k > 1$ be integers.

Then:

$$P[X = m+k \mid X > m] = \frac{P[X = m+k,\ X > m]}{P[X > m]}$$

$$= \frac{P[X = m+k]}{P[X > m]}$$

$$= \frac{(1-p)^{m+k-1} p}{(1-p)^m}$$

$$= (1-p)^{k-1} p$$

$$= P[X = k]$$

Your probability of making the shot in k throws does not depend on how many shots you've already missed.

Problem 9

(a) Each nucleotide can mutate. Let

$$Y_k = \begin{cases} 1 & \text{if the } k^{\text{th}} \text{ nucleotide mutates} \\ 0 & \text{if not} \end{cases}$$

Then $Y_k \sim \text{Bern}(p)$ where $p = 2 \times 10^{-8}$.

And the nucleotide mutations are independent, like coin flips. So

$$X = Y_1 + Y_2 + \cdots + Y_{3,200,000,000}.$$

Thus $X \sim \text{Binom}(n = 3,200,000,000,\ p = 2 \times 10^{-8})$

(b) $E[X] = np = (3,200,000,000)(2 \times 10^{-8}) = 64$

This is μ.

(c) For $\text{Bin}(n,p)$, $\sigma^2 = np(1-p)$.

So $\sigma^2 = \text{Var}(X) = np(1-p)$

$= (3,200,000,000)(2 \times 10^{-8})(1 - 2 \times 10^{-8})$

≈ 63.9999

≈ 64

$\Rightarrow \sigma = 8$.

The interval is $(56, 72)$

(d) $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$. The difficulty is that p is very small and n is enormous.

Problem 9

(e) Law of rare events says Y can approximate X because p is small and n is big. Now, with $\lambda = 64$,

$$P[Y=k] = \frac{e^{-64}\,64^k}{k!}$$

So $P[Y=1] = e^{-64} \cdot 64 \approx 0$

$$P[Y=5] = \frac{e^{-64}\,64^5}{5!} \approx 10^{-20} \approx 0.$$

$P[Y \geq 3] \approx 1$.

etc.

Problem 10

Recall $P[Y=y] = \frac{e^{-\lambda}\lambda^y}{y!}$, $y \in \{0,1,2,\dots\}$.

Thus, $E\left[\frac{1}{1+Y}\right] = \sum_{y=0}^{\infty} \frac{1}{y+1} P[Y=y]$ (def. of Expectation)

$$= \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{(y+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{(y+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} \underbrace{\left(\frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \dots\right)}_{= e^{\lambda} - 1, \text{ by the formula } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

$$= \frac{e^{-\lambda}}{\lambda}\left(e^{\lambda} - 1\right)$$

$$= \frac{1 - e^{-\lambda}}{\lambda}.$$