Midterm 2: Practice Exam 1

Problem 1. An evil sorcerer decides to summon some minions to help clean up his lawn after a big storm. His magic spell summons up to 2 skeletons and up to 2 abyssal specters. Let X be the number of skeletons summoned, and let Y be the number of abyssal specters summoned. According to the notes in his spellbook, the joint pmf of X and Y is given by:

$p(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.12	0.10	0.08
$x=1$	0.06	0.18	0.14
$x=2$	0.04	0.08	0.20

(a) What is the probability that the necromancer summons an even number of minions?

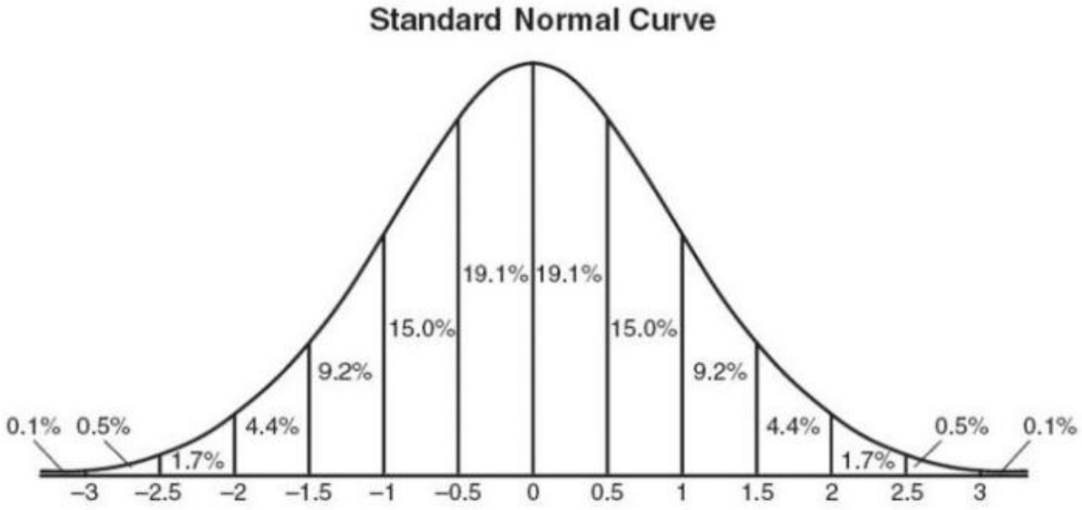
(b) Compute $\mathbb{P}[X \leq 1 \text{ and } Y \leq 1]$.

(c) Using words, describe the event $[X+Y \geq 2]$, and then compute its probability.

(d) Compute the marginal pmf of X and Y. Using $p_X(x)$, find $\mathbb{P}[X \geq 1]$.

(e) Are X and Y independent? Justify your answer.

Problem 2. A machine that produces ball bearings has initially been set so that the true average diameter of the bearings it produces is .500 cm. A bearing is acceptable if its diameter is within .004 cm. of this target value. Suppose, however, that the setting has changed during the course of production, so that the bearings have normally distributed diameters with mean value .499 cm. and standard deviation .002 cm. What percentage of the bearings produced will not be acceptable? [Hint: use the following plot of the standard normal density curve.]



Problem 3. Fix $\beta > 0$, and let X be a random variable with cdf

$$F(t) = \begin{cases} 1 - \left(\frac{\beta}{t}\right)^3 & : \quad t \geq \beta \\ 0 & : \quad t < \beta \end{cases}$$

(a) Find f, the pdf of X.

(b) Find $\mathbb{E}[X]$.

(c) Find $\mathbb{P}[\beta \leq X \leq 2\beta]$.

Problem 4. Let T be the time, in minutes, between successive drink orders at a bar. Suppose that a reasonable probability model for T is given by the pdf

$$f_T(t) = Ce^{-\frac{t}{4}}\mathbf{1}_{[t\geq 0]},$$

where is $C > 0$ is some number that you'll have to determine.

(a) Find a value for C such that $f_T(t)$ is a valid pdf.

(b) What is the probability that the time between successive orders is greater than 4 minutes?

(c) On average, how long must the bartender wait between drink orders?

Problem 5. It is a well-known fact that 1% percent of treasure chests are actually mimics.[1] A wizard has developed a magical detection spell to identify mimics before opening them. However, the spell is imperfect:

$$\mathbb{P}\left[\text{spell says "safe"} \mid \text{chest is a mimic}\right] = 0.03$$

$$\mathbb{P}\left[\text{spell says "mimic"} \mid \text{chest is normal}\right] = 0.05$$

(a) What is the probability that a randomly-selected treasure chest chest is flagged as a mimic?

(b) Suppose a treasure chest is flagged as a mimic. What is the probability that it is actually a mimic?

Problem 6. Let A and B be events for which you know $\mathbb{P}[A \cup B] = 0.8$, $\mathbb{P}[A] = 0.4$, and $\mathbb{P}[A \cap B] = 0.3$.

(a) Find $\mathbb{P}[A]$

(b) Are A and B independent? Justify your answer.

(c) Are A and B disjoint? Justify your answer.

(d) Find $\mathbb{P}[A \cap B^c]$

(e) Find $\mathbb{P}[A \mid A \cup B]$

(f) Find the probability that exactly one of the two events A, B occurs.

Problem 7. Consider a deck of four cards consisting of one card numbered '1', two cards numbered '2', and one card numbered '3'. The deck is suffled thoroughly. Let D denote the difference between the numbers on the top two cards, ignoring the sign.

(a) Display the probability mass function of D in a suitable table.

(b) Find $\mathbb{E}[D]$

(c) Find $\text{Var}(D)$

[1] A *mimic* is a type of monster which disguises itself as a treasure chest and attempts to eat you when you open it.

Midterm 2: Practice Exam 2

Problem 1. Suppose you station yourself at a spot on an idyllic country road and watch as cars pass by. Let T be the waiting time, in minutes, between the first and second cars to pass by. Assume $T \sim \exp(\lambda)$, i.e., that the pdf of T is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & : \quad t \geq 0 \\ 0 & : \quad t < 0. \end{cases}$$

(a) Suppose the rate of cars passing by is $\lambda = 3$ cars per minute. Do a calculation which shows that the average waiting time between cars is 20 seconds.

(b) Show that $\mathbb{P}[T > s + t \mid T > s] = \mathbb{P}[T > t]$ for all $t, s > 0$.

(c) Interpret the equation from Part (b) in words.

Problem 2. A large number of drinking wells are tested for toxic chemical contaminants. Let A be the event that arsenic is detected, let B be the event that benzene is detected, and let C be the event that chromium is detected. Suppose that

$$\mathbb{P}[A] = .4, \qquad \mathbb{P}[B] = .37, \qquad \mathbb{P}[C] = .23, \qquad \mathbb{P}[A \cap B \cap C] = .05,$$
$$\mathbb{P}[A \cap B] = .25, \qquad \mathbb{P}[A \cap C] = .07, \qquad \mathbb{P}[B \cap C] = .1.$$

(a) Draw and label a Venn diagram using the above probabilities.

(b) What proportion of wells have both arsenic and benzene, but not chromium?

(c) A well is randomly selected. What is the probability that it is not contaminated?

(d) Given that arsenic is detected at a drinking well, what is the probability that at least one of the other two chemicals is detected?

Problem 3. Let X be a random variable with finite mean $\mu = \mathbb{E}[X]$. Show that

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mu^2.$$

Problem 4. Let M be the maximum of two rolls of a 6-sided dice.

(a) What is the pmf of M?

(b) Determine the cdf of M and graph it.

(c) What is the expected value of M?

Problem 5. A bank offers two service counters: (1) a drive-up counter, and (2) a walk-in counter. In a random workday, let X be the proportion of time that the drive-up counter is in use, and Y the proportion of time that the walk-in counter is in use. Assume that the joint pdf of (X, Y) is

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & : \quad 0 \leq x, y \leq 1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

(a) Verify that f is a valid pdf.

(b) Find the probability that neither counter is in use more than one-quarter of the time.

(c) Find the marginal density of Y, f_Y.

(d) Using f_Y, compute $\mathbb{P}[.25 \leq Y \leq .75]$.

Problem 6. Suppose the force acting on a column that supports a building is normally distributed with mean value 15 kips and standard deviation 1.25 kips. Compute the following probabilities by standardizing and then inspecting the standard normal curve shown below:

(a) $\mathbb{P}[X \leq 15]$

(b) $\mathbb{P}[X \leq 17.5]$

(c) $\mathbb{P}[X \geq 10]$

(d) $\mathbb{P}[14 \leq X \leq 18]$

(e) $\mathbb{P}[|X - 15| \leq 3]$

