Lecture Notes for Math 243: Calculus III

Last updated: November 4, 2026

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0 Tentative Course Outline

- **Weeks 1-3:** Parametric equations and Polar coordinates
  1. Arc length
  2. Curves defined by parametric equations.
  3. Calculus with plane curves.
  4. Polar coordinates
  5. Areas and lengths in polar coordinates
  6. Conic sections.
- **Weeks 4-6:** Vectors and the geometry of space.
  1. Three-dimensional coordinate systems.
  2. Vectors.
  3. The dot product.
  4. The cross product.
  5. Equations of lines and planes.
  6. Cylinders and quadric surfaces.
- **Friday, October 16 (week 8):** Midterm exam
- **Weeks 7-9:** Vector functions.
  1. Vector functions and space curves
  2. Derivatives and integrals of vector functions.
  3. Arc length and curvature.
  4. Motion in space: velocity and acceleration.
- **Weeks 10-16:** Partial derivatives.
  1. Functions of several variables.
  2. Limits and continuity
  3. Partial derivatives.
  4. Tangent planes and linear approximations
  5. The chain rule.
  6. Directional derivatives and the gradient vector.
  7. Maximum and minimum values.
  8. Discovery project (p.1010 Stewart): Quadratic approximation and critical points (Taylors formula for two variables).
  9. Lagrange multipliers.
- **Monday, December 14:** Final exam.

1 2026-08-24 | Week 01 | Lecture 01

This lecture is based on textbook section 6.3. "Arc length"

> ***The nexus question of this lecture:*** *How do we compute the length of a curve, assuming the curve is the graph of a function?*

1.1 Arc Length

First, recall the definition of ***derivative***:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or equivalently,

$$f'(x) = \lim_{\tilde{x} \to x} \frac{f(\tilde{x}) - f(x)}{\tilde{x} - x}$$

In other words, the derivative is obtained by computing

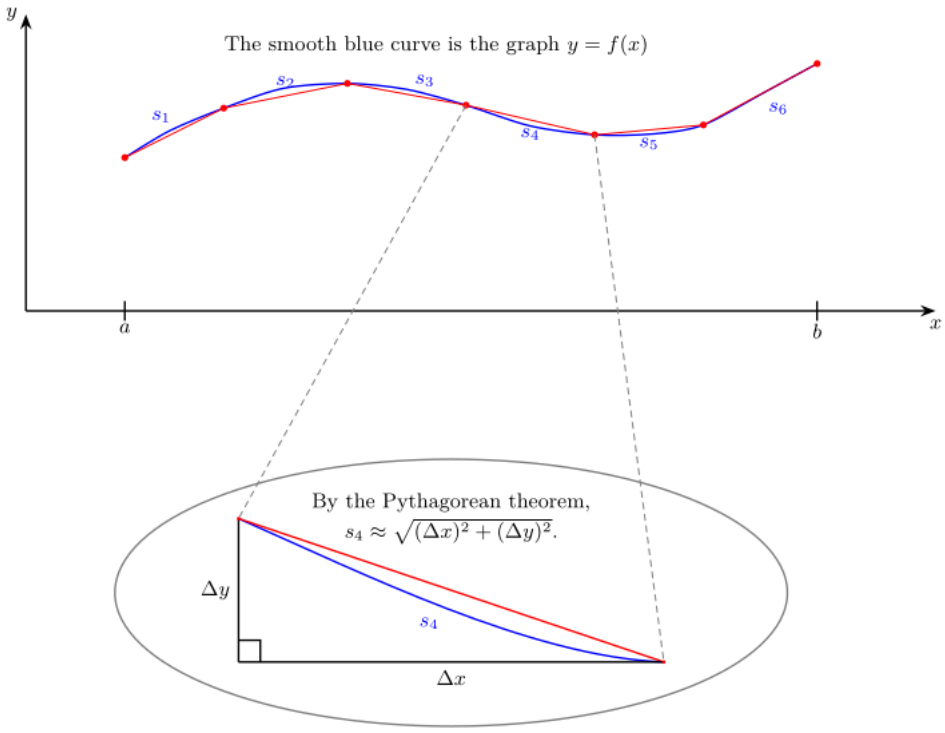
$$\frac{\text{change in } y\text{-value}}{\text{change in } x\text{-value}} = \frac{\Delta y}{\Delta x}$$

as $\Delta x \to 0$.

One of the central ideas in calculus is to approximate curvy things with flat things. For example, if faced with a curve, approximate it with straight lines. For example,

Problem: I am an ant, and I want to travel from a to b by walking on the graph of $f(x)$. How far must I walk?

Strategy: Cheat a little: think of the ant walking in a bunch of straight lines that are pretty close to the curve.



In this example, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ can be decomposed as

$$(\text{Arc Length}) = s_1 + s_2 + \ldots + s_6,$$

where

$$s_i = (\text{length of segment } i)$$

Each segment length can be approximated by a straight line. For example,

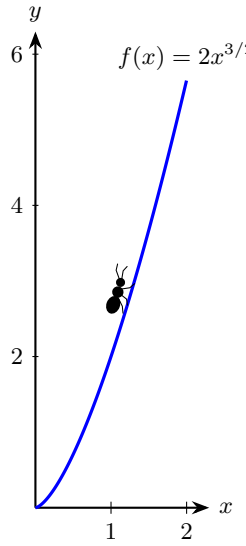
$$\begin{aligned}
s_4 &\approx \sqrt{(\Delta x)^2 + (\Delta y)^2} && \text{since the line segment length approximates the curve length} \\
&= \sqrt{\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right](\Delta x)^2} \\
&= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x \\
&\approx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx && \text{this holds when } \Delta x \approx 0, \text{ so long as the curve is smooth} \\
&= \sqrt{1 + [f'(x)]^2}\,dx && \text{by change of notation}
\end{aligned}$$

This example motivates the following theorem.

Theorem 1 (Arc length formula)**.** *Let f be a differentiable function on the interval $[a, b]$, and assume that f' is continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is*

$$(\textit{Arc Length}) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Example 2 (Arc length)**.** Find the length of the curve defined by $f(x) = 2x^{3/2}$, where x ranges from $x = 0$ to $x = 2$.



Solution: $f'(x) = 3x^{\frac{1}{2}}$, so

$$\begin{aligned}
(\text{Arc length}) &= \int_0^2 \sqrt{1 + [f'(x)]^2}\,dx \\
&= \int_0^2 \sqrt{1 + 9x}\,dx
\end{aligned}$$

which we can solve with a u-substitution, taking $u = 1 + 9x$. Doing this gives a length of about 6.06.

End of Example 2. □

The ***hyperbolic cosine*** and ***hyperbolic sine*** functions are defined as

$$\cosh(x) := \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

These two functions are "nice" in that they behave in some respects similarly to the standard sine and cosine functions; in particular, cosh is an even function (like $\cos(x)$) and $\sinh(x)$ is odd (like $\sin(x)$).

One can check they satisfy:

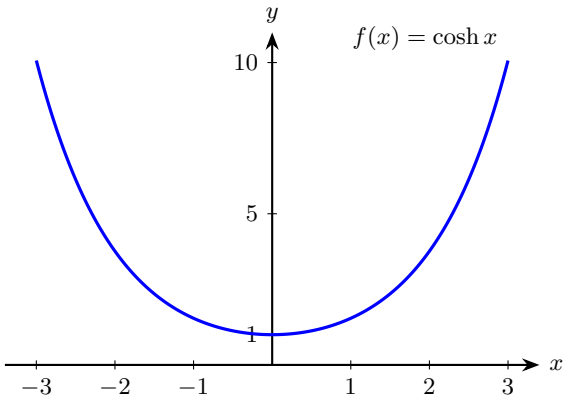
$$\cosh'(x) = \sinh(x), \qquad \sinh'(x) = \cosh(x). \tag{1}$$

Also, we have the following identity:

$$\cosh^2(x) - \sinh^2(x) = 1. \tag{2}$$

A hanging chain forms a curve known as a ***catenary***. The general formula of a catenary is a function involving hyperbolic cosine; in the next example, we'll consider the simplest case.

Example 3 (Arc length)**.** Suppose $f(x) = \cosh(x)$ represents a chain from two endpoints at $x = -3$ and $x = 3$. Find the length of the chain.



$$\begin{aligned}
(\text{length of chain}) &= \int_{-3}^{3} \sqrt{1 + \left[\cosh'(x)\right]^2}\,dx \\
&= \int_{-3}^{3} \sqrt{1 + \sinh^2(x)}\,dx && \text{since } \cosh'(x) = \sinh(x) \text{ by Eq. (1)} \\
&= \int_{-3}^{3} \sqrt{\cosh^2(x)}\,dx && \text{by Eq. (2)} \\
&= \int_{-3}^{3} \cosh(x)\,dx && \text{since } \cosh(x) > 0 \text{ for all } x \in \mathbb{R} \\
&= [\sinh(x)]_{x=-3}^{x=3} && \text{since } \sinh'(x) = \cosh(x) \text{ Eq. (1)} \\
&= \sinh(3) - \sinh(3) \\
&\approx 20.036
\end{aligned}$$

End of Example 3. □

Tips

- Check that your answer is positive. Arc length is never negative.
- These problems are usually contrived so that you can actually do the integral. If the integral looks took hard, double check your work. Also, there might be a trick to the integral.

2 2026-08-26 | Week 01 | Lecture 02

This lecture is based on textbook section 6.3 "Arc Length", and section 11.1 "Parametric Equations and Polar Coordinates". Please read 6.3 and begin start reading 11.1.

Example 4 (Warm up problem: Chain rule practice)**.** Find the derviative of

$$f(x) = \log\left(\frac{x^2}{\sin x}\right).$$

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\log\left(\frac{x^2}{\sin x}\right)\right] \\ &= \frac{\sin x}{x^2}\frac{d}{dx}\left[\frac{x^2}{\sin x}\right] && \text{by the chain rule} \\ &= \frac{\sin x}{x^2}\left(\frac{2x\sin x - x^2\cos x}{(\sin x)^2}\right) && \text{by the quotient rule} \\ &= \frac{2}{x} - \cot x && \text{after simplification.} \end{aligned}$$

End of Example 4. □

2.1 Arc length - examples

Example 5 (Arc length)**. Problem:** Let

$$f(x) = \frac{x^3}{3} + \frac{1}{4x} + 7,$$

for $1 \le x \le 2$. earlier examples.

Solution: We want to apply Theorem 1. To do this, we first compute $1 + [f'(x)]^2$:

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(x^2 - \frac{1}{4}x^{-2}\right)^2 \\ &= x^4 + \frac{1}{2} + \frac{1}{16}x^{-4} \\ &= \left(x^2 + \frac{1}{4}x^{-2}\right)^2. \end{aligned}$$

The last step, where we write the equation as a perfect square, is kinda hard to see. But since we're familiar with Example 3, we know to possibly expect something like that.

Now, we can apply the formula from Theorem 1:

$$\begin{aligned}
\text{Arc Length} &= \int_1^2 \sqrt{1 + [f'(x)]^2}\,dx \\
&= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4}x^{-2}\right)^2}\,dx \\
&= \int_1^2 x^2 + \frac{1}{4}x^{-2}dx \\
&= \left[\frac{x^3}{3} - \frac{1}{4}x^{-1}\right]_1^2 \\
&= \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) \\
&\approx 2.46.
\end{aligned}$$

End of Example 5. □

Example 6. Find the arc length of

$$f(x) = \frac{x^3}{12} + x^{-1}, \qquad 1 \le x \le 4$$

Solution: For the complete solution, see Example 2 in section 6.3 of the textbook. We sketch the approach here.

The key step is to factor

$$\begin{aligned}
1 + [f'(x)]^{-2} &= \frac{x^4}{16} + \frac{1}{2} + x^{-4} \\
&= \left(\frac{x^2}{4} + x^{-2}\right)^2
\end{aligned}$$

The arc length integral then reduces to

$$\int_1^4 \left(\frac{x^2}{4} + x^{-2}\right) dx$$

which evaluates to 6.

End of Example 6. □

2.2 Introduction to parametric equations

This section is based on chapter 11.1 in the textbook.

I am an ant walking around the whiteboard, and I'm covered in ink. My path is described in the following way:

- The variable t measures time. We start at $t = 0$ and end at time $t = T_{\text{end}} > 0$.
- At each moment in time t, the number $x(t)$ represents my x-coordinate
- And the number $y(t)$ represents my y-coordinate.

The ***parametric equation*** is the picture I draw by walking around the whiteboard.

Definition 7 (Parametric Equations and Curves)**.** Suppose x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values I. Then as t varies over I, the set of all points of the form

$$(x, y) = (f(t), g(t)), \tag{3}$$

as t varies over I, is called a ***parametric curve***. The equations in Eq. (3) are called the ***parametric equations*** of the curve.

The variable t is called the ***parameter***. If $I = [a, b]$, then the ***initial point*** is $(f(a), g(a))$ and the ***terminal point*** is $(f(b), g(b))$.

Notation 8. In "set-builder notation," the graph of the parametric equations $x = f(t)$ and $y = g(t)$ is the set

$$\{(f(t), g(t)) : t \in I\}.$$

For those unfamiliar with this notnation: the colon ':' is read as "such that" and 'ϵ' is an abbreviation of "is an element of"

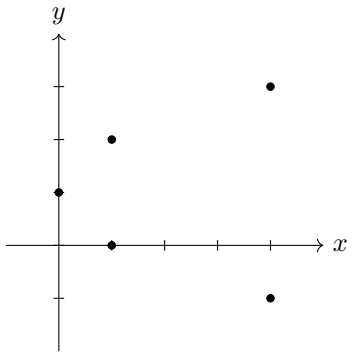
Example 9 (Plotting parametric functions)**.** Plot the graph of the parametric equations

$$\begin{cases} x = t^2 \\ y = t + 1 \end{cases}$$

for t in $I = [-2, 2]$.

Solution: Start by plotting some points by making a table.

t	$x = t^2$	$y = t + 1$
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3



Once we get a sense of what the curve looks like, we can draw a curve through the points.

- If we replace $I = [-2, 2]$ by $I = \mathbb{R}$, then this corresponding graph would trace out a sideways opening parabola defined by the equation $x = (y - 1)^2$.

Question: can we write y a function of x?
Answer: No, because the graph doesn't satisfy the vertical line test.

Question: can we express this as an equation involving only x and y?
Answer: yes, we the parabola is $x = (y - 1)^2$, so the parametric curve is can be written as

$$x = (y - 1)^2, \qquad 0 \le x \le 4.$$

End of Example 9. □

3 2026-08-28 | Week 01 | Lecture 03

This lecture is based on section 11.1 in the textbook

Reading assignment: Section 11.1 in the text book.

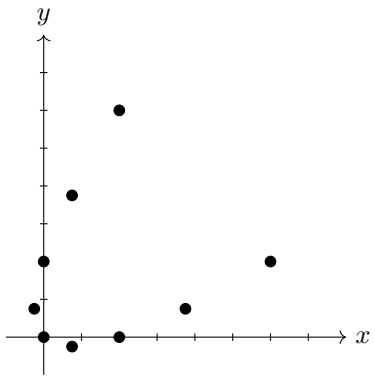
3.1 Plotting parametric equations

Example 10. *Question:* Plot the graph of the parametric equation

$$r(t) = (t^2 + t, t^2 - t), \quad -2 \leq t \leq 2$$

Solution: Again, we plot some points by making a table.

t	$x = t^2 + t$	$y = t^2 - t$
-2	2	6
$-\frac{3}{2}$	.75	3.75
-1	0	2
$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{4}$
0	0	0
$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$
1	2	0
$\frac{3}{2}$	3.75	$\frac{3}{4}$
2	6	2



The graph of the parametric equations is obtained by drawing the curve through the plotted points.

End of Example 10. □

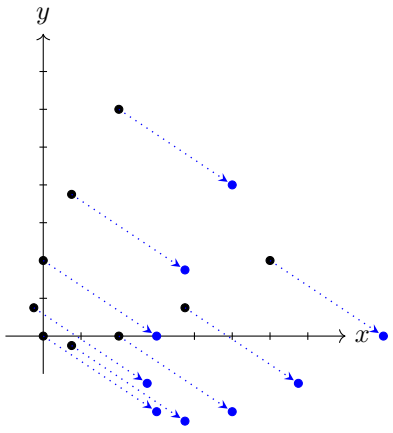
Example 11. Find the parametric equations that shift the graph from Example 10 to the right by 3 and down by 2.

Solution: The graphs of parametric equations are easy to shift. The parameteric equations defining the shifted graph are

$$r(t) = (t^2 + t + 3, t^2 - t - 2)$$

We've simply added 3 to the x-value and subtracted 2 from the y-value. To see why this is the case, observe what happens to the points in the plot from Example 10:

t	$x = t^2 + t+3$	$y = t^2 - t-2$
-2	$2+3$	$6-2$
$-\frac{3}{2}$	$.75+3$	$3.75-2$
-1	$0+3$	$2-2$
$-\frac{1}{2}$	$-\frac{1}{4}+3$	$\frac{3}{4}-2$
0	$0+3$	$0-2$
$\frac{1}{2}$	$\frac{3}{4}+3$	$-\frac{1}{4}-2$
1	$2+3$	$0-2$
$\frac{3}{2}$	$3.75+3$	$\frac{3}{4}-2$
2	$6+3$	$2-2$



Here we have shifted every point to the right by 3 and down by 2, as shown by the arrows. Drawing a curve through the blue points now gives us the curve of the new, shifted, parametric equations. (Not shown in figure).

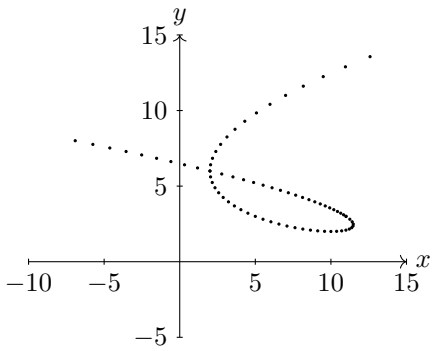
End of Example 11. □

Parametric functions have the advantage that we can use them to plot curves which cannot be plotted by "$y = f(x)$" type graphs.

Example 12. Here's an example of a plot of the parametric curve

$$r(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t^3 - 5t^2 + 3t + 11 \\ t^2 - 2t + 3 \end{pmatrix}$$

This one would be tedious to plot by hand, so instead I plotted a bunch of points with a computer, with t varying on the interval $I = [-1.5, 4.5]$.



The parametric equation can be thought of as tracing the trajectory of a particle as it moves over time. In this example, t represents time. At time $t = -1.5$, the particle is at point $(x, y) = (-8.125, 8.25)$, and starts heading southeast. It then turns around and loops over its previous path, before heading northeast.

It would be impossible to represent this trajectory using a function $y = f(x)$, becausethe curve fails the vertical line test. This is one advantage of parametric curves.

The particle has the same position at times $t = -1$ and $t = 3$. At these times, the particle is at the point

$$(x, y) = (2, 6).$$

Thus, it is at this point that the curve intersects itself, a phenomenon known as a ***singularity***.

End of Example 12. □

3.2 Converting between parametric equations and Cartesian equations

In contrast to parametric equations, an equation is said to be in ***rectangular form*** (aka: ***Cartesian form***) if it takes the form $y = f(x)$ (or, less commonly, $x = f(y)$) for some function f. We've just seen in Example 12 that parametric equations offer significantly more flexibility when graphing.

In fact, it is always possible to convert an equation from rectangular form into parametric form. We illustrate this with the next example:

Example 13 (Converting from rectangular form to parametric form)**.** Consider the equation $y = x^2$, which is in rectangular form. This is an upward-opening parabola. There are many ways to represent this in parametric form. The simplest is with the parametric equations

$$r(t) = (t, t^2), \qquad \infty < t < \infty.$$

End of Example 13. □

Example 14 (Converting from parametric form to rectangular form)**.** Sometimes (but not always) we can also convert from parametric to rectangular form. Consider the parametric equations

$$\begin{cases} x = t^2 \\ y = 7 + 5t \end{cases}$$

This is done by "eliminating" the parameter t, in the following way:

Since $y = 7 + 5t$, it follows that

$$t = \frac{y-7}{5}.$$

Plugging this into $x = t^2$ gives

$$\begin{aligned} x &= \left(\frac{y-7}{5}\right)^2 \\ &= \frac{1}{25}\left(y^2 - 14y + 49\right) \\ &= \frac{1}{25}y^2 - \frac{14}{25}y + \frac{49}{25}. \end{aligned}$$

This equation has the form $x = f(y)$, and therefore is in rectangular (i.e., Cartesian) form.

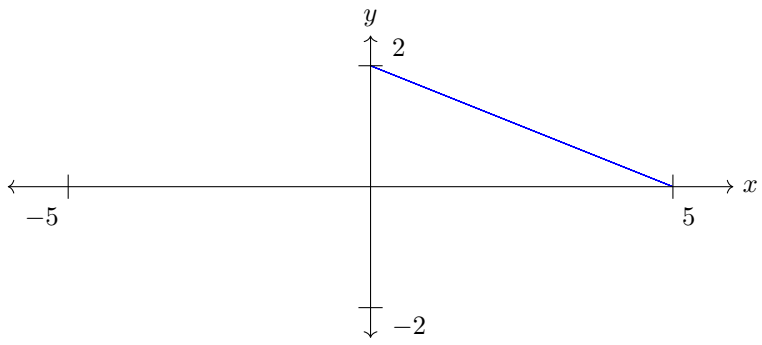
End of Example 14. □

Here's another example

Example 15 (Converting to Cartesian form: eliminating a parameter)**.** Consider the parametric equations

$$\begin{cases} x = 5\cos^2(\theta) \\ y = 2\sin^2(\theta) \end{cases}, \qquad 0 \le \theta \le 2\pi.$$

I plotted this using a computer and got the following. It is a line segment:



Question: Convert these parametric equations into Cartesian form, ie a function of the form $y = f(x)$.

Solution: Before we start, recall the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for all θ. using this, we have:

$$\begin{aligned} y &= 2\sin^2(\theta) \\ &= 2\left(1 - \cos^2(\theta)\right) \end{aligned}$$

Now, $x = 5\cos^2(\theta)$, so $\cos^2(\theta) = \frac{x}{5}$. Plugging this into the above equation gives:

$$y = 2\left(1 - \frac{x}{5}\right).$$

This is an equation of a line.

But we are not done! We are trying to plot only a line *segment*, not the full line. So we need to specify that we are restricting the line to only those x in the interval $[0, 5]$. So our solution (in Cartesian form) is

$$\begin{cases} y = 2\left(1 - \frac{x}{5}\right) \\ 0 \le x \le 5 \end{cases}$$

End of Example 15. □

4 2026-08-31 | Week 02 | Lecture 04

Reading assignment: Chapter 11.2 "Calculus in Parametric Curves"

This is probably the most important lecture I will give this semester, so please try to understand it in detail.

4.1 Ellipses

Definition 16 (Ellipse)**.** An ***ellipse*** centered at (h, k) with horizontal radius $a > 0$ and vertical radius $b > 0$ has equation

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1.$$

Example 17 (Eliminating a parameter)**.** Eliminate the parameter t in the parametric system of equations

$$\begin{cases} x = 4\cos(t) + 3 \\ y = 2\sin(t) + 1 \end{cases} \tag{4}$$

The trig identity $\sin^2(t) + \cos^2(t) = 1$ served us well in Example 15, so let's try to use that. First, solve for $\sin(t)$ and $\cos(t)$:

$$\sin(t) = \frac{y-1}{2}, \qquad \cos(t) = \frac{x-3}{4}. \tag{5}$$

By the trig identity, we have

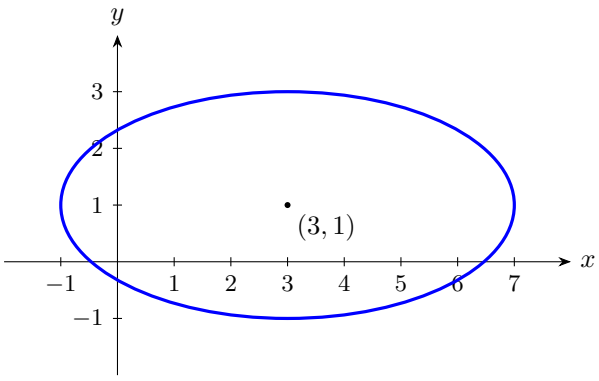
$$\cos^2(t) + \sin^2(t) = 1.$$

Substituting in the formulas from Eq. (5) gives:

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1,$$

This is the equation of an ellipse, with center $(3, 1)$ and with horizontal and vertical axes 4 and 2 respectively. These numbers can be read off of Eq. (4).

The plot is



At time $t = 0$, the ant starts at position $(x, y) = (7, 1)$ and then marches counter-clockwise around the ellipse. At $t = \pi/2$, it reaches the point $(3, 3)$. At time $t = \pi$ it reaches $(-1, 1)$. At time $t = 2\pi$, the ant returns to the point where it started $(7, 1)$.

End of Example 17. □

4.2 Parameterized curves: velocity, speed, and direction

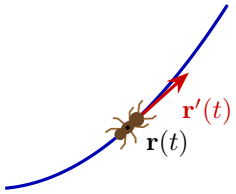
Suppose an ant is walking along the curve in the xy-plane defined by the parametric equation

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad a \le t \le b$$

where t represents time.

Three questions:

- what is the ant's velocity?
- how fast is the ant walking?
- what direction is the ant facing?



Answers:

$$\text{velocity} = \mathbf{r}'(t), \qquad \text{speed} = |\mathbf{r}'(t)|, \qquad \text{direction} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

We'll explain them one-by-one.

Velocity. First, recall that ***velocity*** is defined as the derivative of position with respect to time. The ant's position at time t is $\mathbf{r}(t)$. Therefore its velocity at time t is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}. \tag{6}$$

To see geometrically why this is natural, let $\Delta t > 0$ represent a small amount of time. The ant's position at time t and $t + \Delta t$ are

$$\mathbf{r}(t) \quad \text{and} \quad \mathbf{r}(t + \Delta t).$$

The ant's average velocity between times t and $t + \Delta t$ is

$$(\text{average velocity}) = \frac{\text{displacement}}{\text{length of time}} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

As $\Delta t \to 0$, the average velocity converges to the instantaneous velocity $\mathbf{v}(t)$; the right-hand side converges to $r'(t)$ (by the definition of derivative). Hence

$$\mathbf{r}'(t) = (\text{velocity at time } t).$$

The vector $\mathbf{r}'(t)$ contains two pieces of information: how fast the ant is moving and which way it is moving.

Speed. Recall that the ***magnitude*** of a vector $\mathbf{v}$, denoted $|\mathbf{v}|$ is the length of the vector. That is, if $\mathbf{v} = (v_1, \ldots, v_n)$, then

$$|\mathbf{v}| = \sqrt{v_1^2 + \ldots + v_n^2}.$$

The ***speed*** of a moving object is the magnitude of its velocity. Therefore, the ant's speed at time t is

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}.$$

Direction. We know the velocity points in the direction the ant is traveling. The vector

$$\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

is simply velocity divided by speed (provided the speed is nonzero!). This vector points in the same direction as the velocity and has length 1, so it records the direction of motion without recording the speed. (This is useful for certain applications.)

4.3 Arc length of a parameterized curve

If the ant walks at a constant speed, then

$$\text{(distance traveled)} = \text{(speed)} \times \text{(time)}.$$

If its speed varies, break the journey into many tiny pieces. During each tiny time interval $[t, t+\Delta t]$, her speed is nearly constant, so

$$\text{(distance traveled from time } t \text{ to } t+\Delta t) \approx \underbrace{|\mathbf{r}'(t)|}_{\text{speed at } t} \times \Delta t.$$

Formally, if the ant starts at time $t = a$ and ends at time $t = b$, we break the time interval into n pieces:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

and put

$$\Delta t_i = t_{i+1} - t_i.$$

Then

$$\text{(Approx. distance traveled)} = \sum_{i=0}^{n-1} |\mathbf{r}'(t_i)| \Delta t_i$$

Geometrically, each term $|\mathbf{r}'(t_i)|\Delta t_i$ is the distance the ant would travel if it moved in a straight line, at constant speed $|\mathbf{r}'(t_i)|$, for Δt_i units of time. Stringing these segments together gives a path made of straight pieces which approximates the curve, in the same spirit as the polygonal approximation in Section 1.1.

As $n \to \infty$ and the time intervals get smaller (i.e., as $\max_i \Delta t_i \to 0$), the approximation gets better and better, and the right hand side (which is a Riemann sum) converges to an integral:

$$\text{(Distance traveled)} = \int_a^b |\mathbf{r}'(t)| dt.$$

Since $|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$, we obtain the following theorem:

Theorem 18 (Arc Length of a Parameterized Curve)**.** *Consider the parameterized curve*

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad a \le t \le b.$$

Then the distance traveled along the curve is

$$\text{(Distance)} = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt, \tag{7}$$

provided that the curve is smooth.

Further, if the parameterization traces the curve only once, this is the arc length of the curve.

Remark 19. *The integrals in Eq.* (7) *are typically difficult to compute by hand. The meaning of smooth will be discussed later.*

End of Remark 19. □

5 2026-09-02 | Week 02 | Lecture 05

Reading assignmnt: Section 11.2 in the textbook.

5.1 Arc length and Distance Examples

Example 20 (Arc length of a circle)**.** Consider the circle parameterized by the equationss

$$\mathbf{r}(t) = \begin{bmatrix} 3\cos(t) \\ 3\sin(t) \end{bmatrix}, \qquad 0 \le t \le 2\pi.$$

What is the arc length?

Solution: The velocity of an ant traversing the curve is

$$\mathbf{r}'(t) = \begin{bmatrix} -3\sin(t) \\ 3\cos(t) \end{bmatrix}.$$

Therefore the speed is

$$|\mathbf{r}'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = \sqrt{9[\sin^2(t) + \cos^2(t)]} = 3,$$

where we have used the identity $\sin^2(t) + \cos^2(t) = 1$.

By Theorem 18, integrating the speed over time gives distance traveled:

$$\begin{aligned} (\text{distance}) &= \int_0^{2\pi} |\mathbf{r}'(t)|\, dt \\ &= \int_0^{2\pi} 3\, dt \\ &= 6\pi. \end{aligned}$$

Since the ant walks around the circle only once in the time interval $[0, 2\pi]$, the distance it traveled is the arc length.

Taking a step back. What was this question really asking? It was asking for the circumference of a circle with radius 3 (since that's the curve traced out by the equations). Using the formula

$$\text{circumference} = 2\pi(\text{radius}),$$

our answer 6π makes sense.

End of Example 20. □

Example 21 (Arc length)**.** What is the arc length of the curve parameterized by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{7}{3}t^3 + t \\ \frac{2}{3}\left(2t^2 + \frac{1}{4}\right)^{3/2} \end{bmatrix}, \qquad 0 \le t \le 2.$$

Solution:

Step 1. Compute $x'(t)$ and $y'(t)$:

$$x'(t) = 7t^2 + 1 \quad \text{and} \quad y'(t) = \left(2t^2 + \frac{1}{4}\right)^{\frac{1}{2}} \cdot 4t.$$

Step 2. Compute $[x'(t)]^2 + [y'(t)]^2$ and simplify.

$$\begin{aligned} [x'(t)]^2 + [y'(t)]^2 &= \left[7t^2 + 1\right]^2 + \left[\left(2t^2 + \frac{1}{4}\right)^{\frac{1}{2}} \cdot 4t\right]^2 \\ &= 49t^4 + 14t^2 + 1 + \left(2t^2 + \frac{1}{4}\right) \cdot 16t^2 \\ &= 49t^4 + 14t^2 + 1 + 32t^4 + 4t^2 \\ &= 81t^4 + 18t^2 + 1. \end{aligned}$$

Step 3. Recognize that you've got a perfect square:

$$81t^4 + 18t^2 + 1 = \left(9t^2 + 1\right)^2$$

(Step 3 doesn't always happen, but it's really nice when it does!)
Step 4. Plug what you get back into the formula from Theorem 18:

$$\begin{aligned}
\text{Arc Length} &= \int_0^2 \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}dt \\
&= \int_0^2 \sqrt{\left(9t^2 + 1\right)^2}dt \\
&= \int_0^2 (9t^2 + 1)dt \\
&= \left[3t^3 + t\right]_0^2 \\
&= 26.
\end{aligned}$$

We conclude that the arc length is 26.

End of Example 21. □

6 2026-09-04 | Week 02 | Lecture 06

Reading assignment: Chapter 11.3

6.1 Smoothness of parameterized curves

Definition 22 (C^1 function)**.** A function f is said to be ***continuously differentiable*** on the interval $[a, b]$ if f is differentiable on $[a, b]$ and f' is continuous. As shorthand, we write

$$f \in C^1[a, b].$$

Definition 23 (Smooth)**.** A curve C parameterized by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad a \leq t \leq b$$

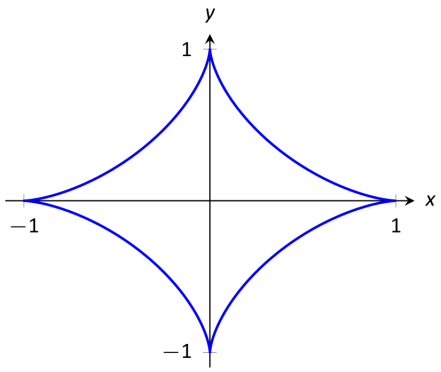
is ***smooth*** if $x, y \in C^1([a, b])$ and $x'(t)$ and $y'(t)$ are never simultaneously zero.

A curve is ***piecewise smooth*** on $[a, b]$ if it can be partitioned into subintervals such that C is smooth on each subinterval.

Example 24 (A non-smooth curve)**.** Consider the equations

$$\begin{cases} x(t) := \cos^3(t) \\ y(t) := \sin^3(t) \end{cases} \qquad 0 \leq t \leq 2\pi.$$

This has graph



Astroid
$x = \cos^3 t$
$y = \sin^3 t$

This sure doesn't look smooth, because it has pointy cusps! Let's show that using the definition.

First, differentiate:

$$\begin{cases} x'(t) = -3\cos^2(t)\sin(t) \\ y'(t) = 3\sin^2(t)\cos(t) \end{cases}$$

Are these functions ever simultaneously zero? Yes, when t takes one of the following values:

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad \text{or} \quad 2\pi.$$

Therefore the curve is not smooth (although it is piecewise smooth).

End of Example 24. □

6.2 Parameterized curves: tangent line and normal line

Let's change notation, and suppose the ant is walking along the curve C parameterized by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad a \le t \le b.$$

At any given instant the ant is facing some direction—the direction of $\mathbf{r}'(t)$—and we can draw a line in that direction. The derivative $\frac{dy}{dx}$ is the *slope of that line.*

How can we find $\frac{dy}{dx}$? The answer to this question is the following important idea:

$$\frac{dy}{dx}(t) = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{8}$$

This idea so important I've put it in a box.

Equation (8) is valid provided that the following two assumptions hold:

- x and y are both differentiable (otherwise $x'(t)$ and $y'(t)$ wouldn't exist)
- $x'(t) \neq 0$ (otherwise we'd have divsion by zero)

Definition 25 (Tangent Line and Normal Line)**.** The ***tangent line*** to C at time $t = t_0$ is the line

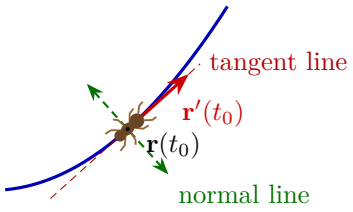
$$y - y(t_0) = \frac{y'(t_0)}{x'(t_0)}\,(x - x(t_0))\,,$$

provided that $x'(t_0) \neq 0$.

The ***normal line*** to C at time $t = t_0$ is the line

$$y - y(t_0) = -\frac{x'(t_0)}{y'(t_0)}\,(x - x(t_0))\,,$$

provided that $y'(t_0) \neq 0$.



Remark 26. *Both the tangent line and the normal line pass through the point $(x, y) = (x(t), y(t))$, which lies on the curve. The only difference is their slope. The tangent line has slope*

$$m = \frac{dy}{dx}(t_0)$$

while the normal line has slope

$$m_{\text{norm}} = -\frac{1}{m}.$$

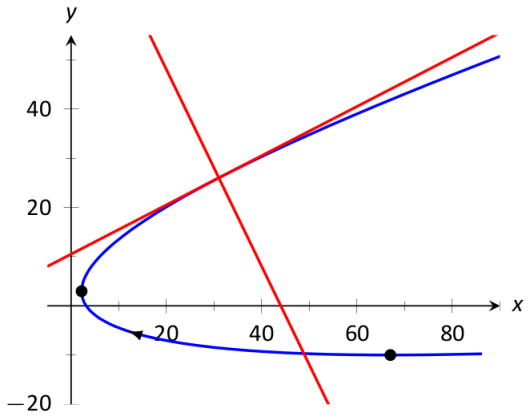
The two lines are perpendicular to each other: the tangent line lies tangent to the curve (parallel to the the velocity $\mathbf{r}'(t_0)$), while the normal line is perpendicular to it.

End of Remark 26. □

Example 27 (Tangent and normal lines)**.** Let C be the curve defined by

$$\begin{cases} x(t) = 5t^2 - 6t + 4 \\ y(t) = t^2 + 6t - 1 \end{cases} \tag{9}$$

Problem: Find the tangent line and normal line to C at $t = 3$.
Solution: Before we start, this is the picture:



Differentiate:

$$\begin{cases} x'(t) = 10t - t \\ y'(t) = 2t + 6 \end{cases}$$

Therefore

$$\frac{dy}{dx}(t) = \frac{2t+6}{10t-6}. \tag{10}$$

Important note: $\frac{dy}{dx}$ is a function of t. It is not a function of x!

At the point $t = 3$, we have:

- $(x, y) = (31, 26)$ (by Eq. (9))
- The slope is $\frac{dy}{dx}(3) = \frac{1}{2}$ (by Eq. (10))

Combining these two facts, and using Definition 25, we conclude that the tangent line is

$$y - 26 = \frac{1}{2}(x - 31)$$

(this is the equation of a line in point-slope form). Here, we have $m_{\tan} = \frac{1}{2}$. The slope of the normal line is always $-\frac{1}{m_{\tan}}$, which is -2. We conclude that the normal line is

$$y - 26 = -2(x - 31).$$

End of Example 27. □

7 2026-09-09 | Week 03 | Lecture 07

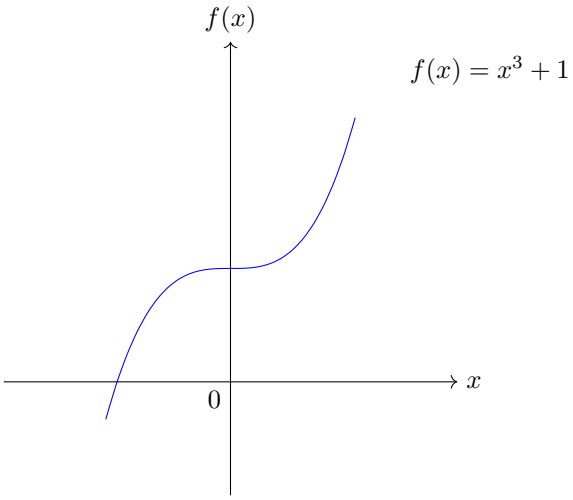
7.1 Concavity of parameterized curves

The concavity of a differentiable function is determined by its second derivative. For example, the graph

$$f(x) = x^2$$

is concave up. Moreover, $f''(x) = 2 > 0$.

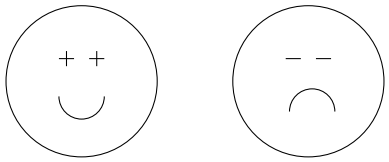
What about $f(x) = x^3 + 1$? This has graph



Tihs graph is concave down when $x < 0$ and concave up when $x > 0$. The concavity is determined by whether the second derivative of f is positive or negative:

$$f''(x) = 6x = \begin{cases} \text{positive} & : \quad x > 0 \\ \text{negative} & : \quad x < 0 \end{cases}$$

This is a general property of twice-differentiable functions: when $f''(x) < 0$, the graph is concave down. When $f''(x) > 0$, the graph is concave up. Just like in the following picture:



With parameterized curves, the story is no different, but we'll need to learn how to compute the second derivative $\frac{d^2y}{dx^2}$, as we now show.

Consider a curve parameterized by

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad a \leq t \leq b.$$

assume that $x(t)$ and $y(t)$ are twice differentiable on the interval $[a, b]$ that $x'(t) \neq 0$ for all $t \in [a, b]$.

We have already seen how to compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

What about $\frac{d^2y}{dx^2}$? What is it?

$$\frac{d^2y}{dx^2}(t) = \text{(instantantous rate of change of slope with respect to } x \text{ at time } t)$$

How to compute it? Let

$$m(t) = \frac{dy}{dx}(t) = \text{(slope of tangent line at time } t)$$

Let dt denote an infinitesimal moment of time. Then

$$\begin{aligned} dm &= m'(t)dt \qquad \text{(infinitesimal change in slope from time } t \text{ to } t+dt) \\ dx &= x'(t)dt \qquad \text{(infinitesimal change in } x \text{ from time } t \text{ to } t+dt) \end{aligned}$$

Then

$$\frac{d^2y}{dx^2} = \frac{dm}{dx} = \frac{m'(t)dt}{x'(t)dt} = \frac{m'(t)}{x'(t)} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)}$$

This give the following formula:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} \tag{11}$$

Example 28 (Concavity of a parameterized curve)**.** Let C be the curve defined by the parametric equations

$$\begin{cases} x(t) = \frac{1}{2}t^2 + \frac{1}{2} \\ y(t) = t - t^3 \end{cases}$$

(a) Find the tangent line to C at $t = 1$.

(b) For what values of t is the curve convex up? For what values is it concave down?

(c) At what points (x, y) does the curve have a horizontal tangent line? Vertical tangent line?

Solution to (a): To apply the line formula $y - y_0 = m(x - x_0)$, we need two things: (1) the point $(x(1), y(1))$ and the slope at $t = 1$.

First observe that

$$(x(1), y(1)) = (1, 0)$$

Second, to compute the slope:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{\frac{d}{dt}\left[t - t^3\right]}{\frac{d}{dt}\left[\frac{1}{2}t^2 + \frac{1}{2}\right]} \\ &= \frac{1 - 3t^2}{t} \end{aligned}$$

So when $t = 1$, we have

$$\frac{dy}{dx}(t) = \frac{dy}{dx}(1) = -2$$

Therefore by the line formula $y - y_0 = m(x - x_0)$, we conclude that the tangent line at $t = 1$ has equation

$$y - 0 = -2(x - 1)$$

or equivalently,

$$y = -2x + 2.$$

Solution to (b). Here, our approach is to compute the second derivatve and then check when it is positive or negative, as this is sufficient to determine whether the curve is concave up or concave down.

To compute the second derivative we'll use the formula:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}.$$

Now, in part (a) we've already computed some components of the above equation. Specifically, we know that $\frac{dy}{dx} = \frac{1-3t^2}{t}$ and $\frac{dx}{dt} = t$. Plugging these in, we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{1-3t^2}{t}\right]}{t}$$

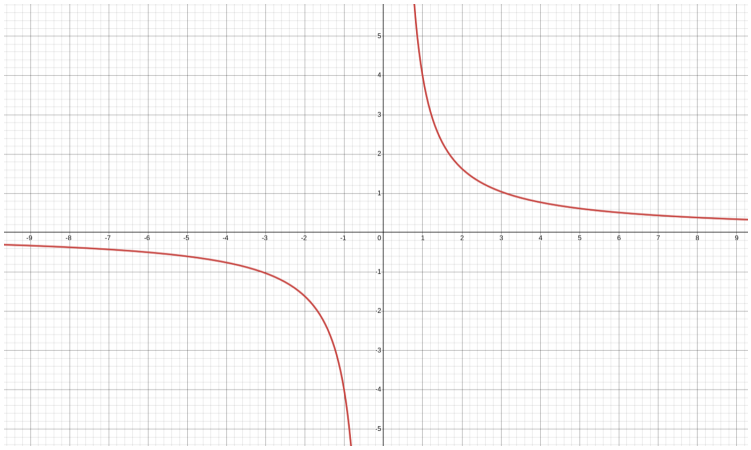
doing the derivative (e.g., using the quotient rule) and simplifying gives

$$\frac{d^2y}{dx^2} = -\frac{1+3t^2}{t^3}$$

One can check that this function is NEGATIVE when $t < 0$, and POSITIVE when $t > 0$. To see this, we can either solve the inequality

$$-\frac{1+3t^2}{t^3} > 0$$

or we can look at the graph of the second derivative:



Solution to (c). To find the horizontal tangent lines, we need check when $\frac{dy}{dx}(t) = 0$. To find the vertical tangent lines, we check when $\frac{dy}{dx}(t) = \pm\infty$ or undefined.

In part (a) we computed

$$\frac{dy}{dx} = \frac{1-3t^2}{t}. \tag{12}$$

By inspection, this is undefined when $t = 0$, since there is a division by zero in that case. Therefore the tangent line is vertical when $t = 0$, which occurs at the point

$$(x(0), y(0)) = \left(\frac{1}{2}, 0\right).$$

On the other hand, we see from Eq. (12) that $\frac{dy}{dx}(t) = 0$ (i.e., the tangent line is horizontal) if and only if

$$1 - 3t^2 = 0$$

or equivalently, when $t = \pm\frac{1}{\sqrt{3}}$. At what points does this occur?

- If $t = \frac{1}{\sqrt{3}}$, then the point is
$$\left(x\left(\frac{1}{\sqrt{3}}\right), y\left(\frac{1}{\sqrt{3}}\right)\right) = \left(\frac{2}{3}, \frac{2}{3\sqrt{3}}\right)$$
- If $t = -\frac{1}{\sqrt{3}}$, then the point is
$$\left(x\left(-\frac{1}{\sqrt{3}}\right), y\left(-\frac{1}{\sqrt{3}}\right)\right) = \left(\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$$

We conclude that the curve C has a horizontal tangent line at the points $\left(\frac{2}{3}, \frac{2}{3\sqrt{3}}\right)$ and $\left(\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$, and a vertical tangent line at the point $\left(\frac{1}{2}, 0\right)$. This answers part (c).

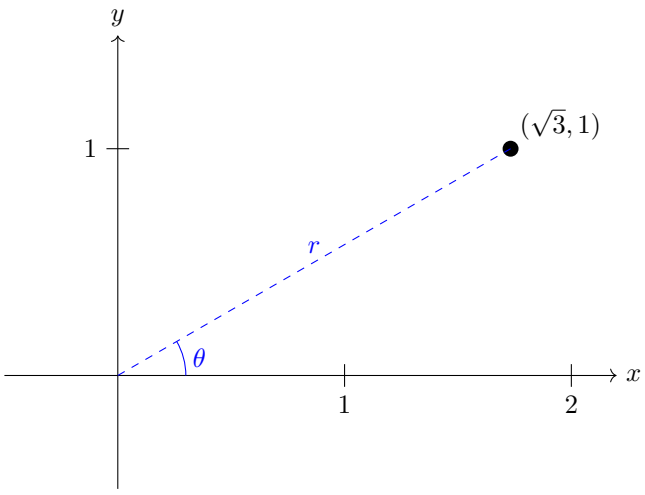
End of Example 28. □

8 2026-09-11 | Week 03 | Lecture 08

Topic of this lecture: *Introduction to polar coordinates*
Reading assignment: *sections 11.3, 11.4*

8.1 Polar Coordinates

Consider the point on the plane whose Cartesian coordinates are $(x, y) = (\sqrt{3}, 1)$:



Here I have also plotted a ray of length r extending from the origin to the point, as well as the angle θ of the ray.

Can we figure out what r and θ are?

- By the Pythagorean theorem, we have

$$r^2 = \left(\sqrt{3}\right)^2 + (1)^2 = 4$$

so $r = 2$.

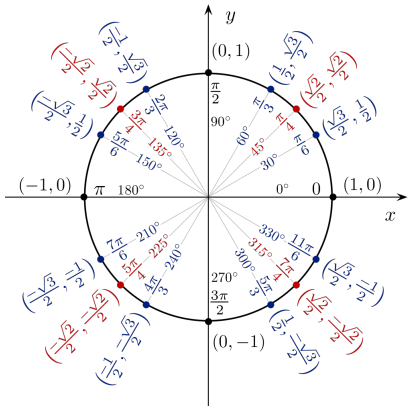
- Observe that $\tan(\theta) = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{1}{\sqrt{3}}$.

And with a little bit of thought after looking at a unit circle, we conclude that

$$\theta = \frac{\pi}{6}$$

since

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}.$$



Therefore, using the two bullet points above, we can represent the point $(x, y) = (\sqrt{3}, 1)$ by the pair

$$(r, \theta) = \left(2, \frac{\pi}{6}\right).$$

This representation is called the ***polar coordinates*** of the point, and it can be done for any point in the xy-plane.

Theorem 29 (Key identities for converting between polar and rectangular coordinates)**.** *If (r, θ) is a point in polar coordinates, the rectangular coordinates are given by*

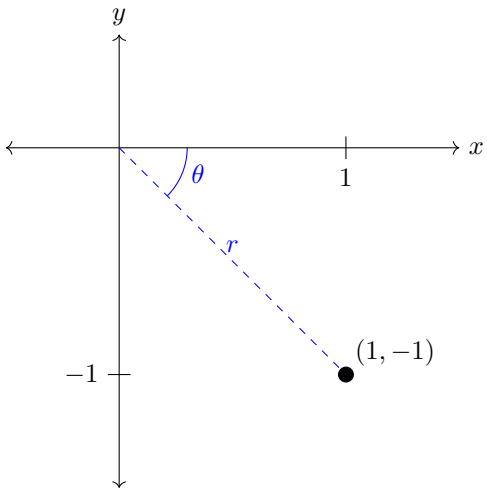
$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta \tag{13}$$

On the other hand, given a point (x, y) in rectangular coordinates, the polar coordinates are determined by the identities

$$r^2 = x^2 + y^2 \quad \textit{and} \quad \tan\theta = \frac{y}{x} \tag{14}$$

Eq. (13) allows us to find the Cartesian coordinates of a point when the polar coordinates are known. More generally, can use the identities in Theorem 29 to convert between coordinate systems, as shown in the next examples.

Example 30. Consider the point on the plane whose Cartesian coordinates are $(1, -1)$. What are the polar coordinates of the point?



Solution: We need to determine r and θ. We use Eq. (14):

- $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.
- $\tan\theta = \frac{y}{x} = -1$. This implies that $\theta = -\frac{\pi}{4}$ (or $\theta = \frac{7\pi}{4}$ works too).

End of Example 30. □

Example 31.

- What is the curve represented by the polar equation $r = 2$?
- What abou the the curve $\theta = \frac{\pi}{3}$?

End of Example 31. □

Example 32. Find a polar equation of the form $r = f(\theta)$ for the curve represented by the Cartesian equation $x = 3$.

Solution: Use $x = r\cos(\theta)$:

$$r\cos(\theta) = 3$$

so

$$r = \frac{3}{\cos(\theta)}$$

or equivalently,

$$r = 3\sec(\theta)$$

End of Example 32. □

9 2026-09-14 | Week 04 | Lecture 09

Topics of this lecture: *Converting between polar and cartesian form, sketching polar curves*

9.1 Converting between Polar to Cartesian form

Example 33 (Cartesian $\implies$ Polar)**.** Find a polar equation of the form $r = f(\theta)$ for the curve represented by the Cartesian equation $x = -2y^2$.

Solution: Use $x = r\cos(\theta)$ and $y = r\sin(\theta)$:

$$\begin{aligned} r\cos(\theta) &= -2\left(r\sin(\theta)\right)^2 \\ &= -2r^2\sin^2(\theta). \end{aligned}$$

Divide both sides by r:

$$\cos(\theta) = -2r\sin^2(\theta).$$

Solve for r:

$$r = -\frac{\cos(\theta)}{2\sin^2(\theta)}$$

End of Example 33. □

Example 34 (Polar $\implies$ Cartesian)**.** Find a Cartesian equation for the polar curve $r = 2\cos(\theta)$.

Solution: By Eq. (13), we know that

$$x = r\cos(\theta).$$

Multiply both sides of our original equation by r:

$$r^2 = 2r\cos(\theta)$$

Thus

$$r^2 = 2x.$$

By Eq. (14), we know that $r^2 = x^2 + y^2$. Therefore

$$x^2 + y^2 = 2x.$$

Completing the square gives

$$(x-1)^2 + y^2 = 1$$

This is a circle of radius 1 with center at $(1, 0)$.

End of Example 34. □

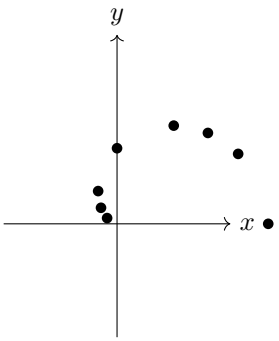
9.2 Sketching polar functions

Example 35. Sketch the polar function

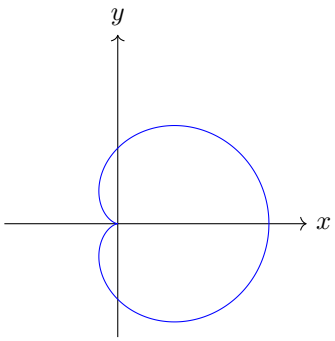
$$\begin{cases} r = 1 + \cos(\theta) \\ 0 \le \theta \le 2\pi \end{cases}$$

Solution: Let's use the method of making a table to plot points. It will be convenient to use the approximations $\sqrt{3} \approx 1.7$ and $\sqrt{2} \approx 1.4$.

θ	$r = 1 + \cos(\theta)$
0	2
$\pi/6$	$1 + \frac{\sqrt{3}}{2} \approx 1.85$
$\pi/4$	$1 + \frac{\sqrt{2}}{2} \approx 1.7$
$\pi/3$	1.5
$\pi/2$	1
$2\pi/3$	$1 - \frac{1}{2} = .5$
$3\pi/4$	$1 - \frac{\sqrt{2}}{2} \approx 1 - 0.7 = .3$
$5\pi/6$	$1 - \frac{\sqrt{3}}{2} \approx 1 - .85 = .15$
π	0



After plotting these points, we make the observation that since $\cos(\theta)$ is an even function, $\cos(-\theta) = \cos(\theta)$. Therefore the graph is symmetric about the horizontal axis. Using this property to sketch the curve, we obtain:



This shape is called a cardiod.

End of Example 35. □

10 2026-09-16 | Week 04 | Lecture 10

Topic of this lecture: *Calculus with polar curves*
Reading assignment: *sections 11.4, 11.5*

10.1 Finding $\frac{dy}{dx}$ with polar functions

No matter what coordinate system we use, the slope of a tangent line is always given by the quantity $\frac{dy}{dx}$.

We've seen how to compute that in other contexts. How do we compute $\frac{dy}{dx}$ when our equation is a polar function of the form $r = f(\theta)$?

As a function of θ, the (x, y)-coordinates are

$$x = f(\theta)\cos(\theta), \quad y = f(\theta)\sin(\theta).$$

By the product rule,

$$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) \quad \text{and} \quad \frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta).$$

Certainly, we must have:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}$$

10.2 Derivatives of polar functions

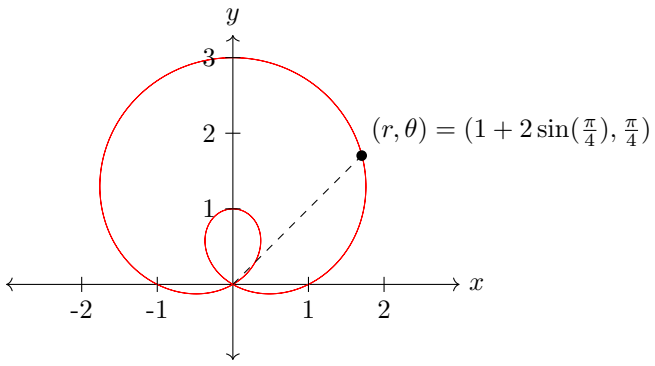
Recall that for a polar curve in the form $r = f(\theta)$, the derivative is

$$\boxed{\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}. \tag{15}$$

Example 36. Consider the polar function

$$r = 1 + 2\sin\theta.$$

This polar function has graph:



Question 1. Find the equations of the tangent and normal lines to the graph at $\theta = \frac{\pi}{4}$.
Solution: To find the tangent line, first compute $\frac{dy}{dx}$. Since $r(\theta) = 1 + 2\sin(\theta)$, we have

$$r'(\theta) = 2\cos(\theta).$$

Therefore, Eq. (15),

$$\begin{aligned}\frac{dy}{dx} &= \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)} \\ &= \frac{2\sin(\theta)\cos(\theta) + (1 + 2\sin\theta)\cos(\theta)}{2\cos^2(\theta) - (1 + 2\sin\theta)\sin\theta}\end{aligned}$$

Plugging in $\theta = \frac{\pi}{4}$, we get

$$\frac{dy}{dx}(\pi/4) = -2\sqrt{2} - 1 \approx -3.8$$

Therefore the slope of tha tangent line is -3.8.
Question 2. For what values of θ is the slope of the tangent line zero? For what values is the tangent line vertical?
Solution sketch: We computed

$$\frac{dy}{dx} = \frac{2\sin(\theta)\cos(\theta) + (1 + 2\sin\theta)\cos(\theta)}{2\cos^2(\theta) - (1 + 2\sin\theta)\sin\theta}$$

To find when the tangent line has slope zero, find the values of θ that make this equation zero.

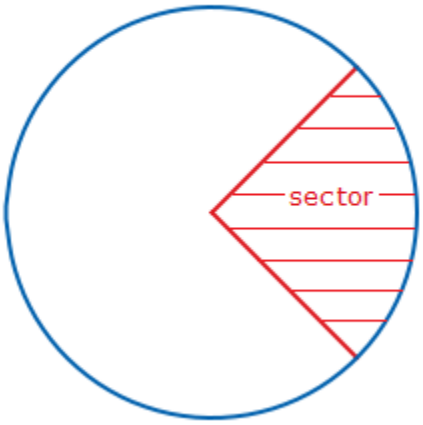
Similarly, a tangent line is vertical when the slope is undefined. So for the second part of the problem, find the values of θ such that the denominator is zero and the numerator is nonzero.

End of Example 36. □

10.3 Areas enclosed by polar graphs

How do we compute the area enclosed by a polar graphs?

A ***sector*** is the technical term for a pie slice of a circle, like the following:



Theorem 37 (Area of a sector)**.** *The area of a sector with angle θ and radius r is*

$$\textit{area of sector} = \frac{1}{2}\theta r^2$$

For example, if $\theta = 360^\circ = 2\pi$, then

$$\text{area } = \pi r^2,$$

which is the formula for the area of a circle.

In class, we derived the following theorem:

Theorem 38. *If $r = f(\theta)$ and f is a continuous and **nonnegative** function on the interval $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$ then the area of the region bounded by the lines $r = f(\theta)$ and $\theta = a$ and $\theta = b$ is*

$$Area = \frac{1}{2}\int_\alpha^\beta r^2 d\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta$$

Remark 39. *It's really important that you check that the conditions $0 \leq \beta - \alpha \leq 2\pi$ and nonnegativity of f. Otherwise cancelletion or double counting of regions may occur, resulting in the integral not actually equalling the area.*

End of Remark 39. □

Throughout we will use the following useful proposition.

Proposition 40 (Useful integrals)**.** *It holds that*

$$\int \cos^2(\theta) d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$
$$\int \sin^2(\theta) d\theta = \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C$$

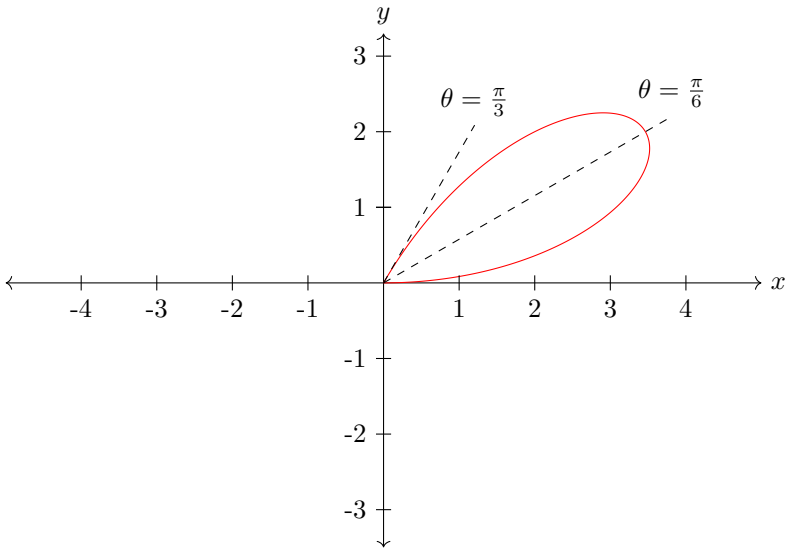
The integrals of Proposition 44 can be computed using the **power-reduction identities** and

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \quad \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}.$$

Example 41 (Area of region enclosed by polar coordinates)**.** Consider the polar function

$$\begin{cases} r = 4\sin(3\theta) \\ 0 \leq \theta \leq \frac{\pi}{3} \end{cases}$$

This is a petal. As θ ranges from 0 to $\frac{\pi}{3}$, the curve loops counterclockwise from the origin back to the origin. To see why, think about the motion of a particle whose trajectory is traced out by $(r, \theta) = (\sin(3\theta), \theta)$. At $\theta = 0$, the particle starts at the origin. When $\theta = \frac{\pi}{6}$, the particle's distance from the origin is maximized. When $\theta = \frac{\pi}{3}$, the particle is back at the origin. This traces out exactly the full petal.. So we will integrate over the interval $0 \leq \theta \leq \frac{\pi}{3}$.



$$\begin{aligned}\text{Area of one petal} &= \frac{1}{2}\int_0^{\pi/3} r^2 d\theta \\ &= \frac{1}{2}\int_0^{\pi/3} \left[4\sin(3\theta)\right]^2 d\theta \\ &= 8\int_0^{\pi/3} \sin^2(3\theta) d\theta.\end{aligned}$$

Use the u-substitution $u = 3\theta$, $du = 3d\theta$

$$\begin{aligned}\text{Area of one petal} &= \frac{8}{3}\int_0^{\pi} \sin^2(u) du \\ &= \frac{8}{3}\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]_{u=0}^{u=\pi} && \text{by Proposition 44} \\ &= \frac{4}{3}\pi.\end{aligned}$$

End of Example 41. □

11 2026-09-18 | Week 04 | Lecture 11

Topic of this lecture: *Computing areas of regions enclosed by polar curves*
Reading assignment: *sections 11.5, start chapter 12.1*

11.1 Computing areas enclosed by polar curves

Theorem 42. *If $r = f(\theta)$ and f is a continuous and **nonnegative** function on the interval $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$ then the area of the region bounded by the lines $r = f(\theta)$ and $\theta = a$ and $\theta = b$ is*

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

Remark 43. *It's really important that you check that the conditions $0 \leq \beta - \alpha \leq 2\pi$ and nonnegativity of f. Otherwise cancelletion or double counting of regions may occur, resulting in the integral not actually equalling the area.*

End of Remark 43. □

Throughout we will use the following useful proposition.

Proposition 44 (Useful integrals)**.** *It holds that*

$$\int \cos^2(\theta) d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$
$$\int \sin^2(\theta) d\theta = \frac{\theta}{2} - \frac{\sin(2\theta)}{4} + C$$

The integrals of Proposition 44 can be computed using the **power-reduction identities** and

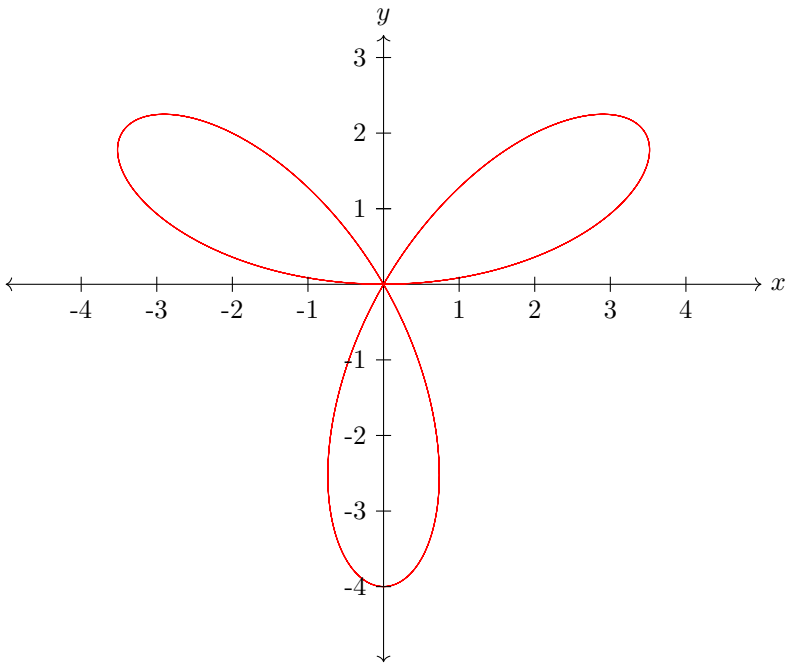
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2} \quad \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}.$$

The next example illustrates by the nonnegativity assumption of Theorem 42 is so essential

Example 45 (Necessity of the hypotheses of Theorem 42)**.** Consider the polar function

$$\begin{cases} r = 4\sin(3\theta) \\ 0 \leq \theta \leq 2\pi \end{cases}$$

This is a continuation of the curve from Example 41, but now we allow θ to range all the way to 2π. This gives the following 3-petal flower:



What is the area of the entire bounded region enclosed by the graph?

The first thing we might think to do is to integrate over $0 \leq \theta \leq 2\pi$, as follows:

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} \left[4\sin(3\theta)\right]^2 d\theta = 8\int_0^{2\pi} \sin^2(3\theta) d\theta.$$

Now do the u-substitution $u = 3\theta$, $du = 3d\theta$:

$$A = \frac{8}{3}\int_0^{6\pi} \sin^2(u) du = \frac{8}{3}\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]_0^{6\pi} = \frac{8}{3}\left[3\pi\right] = 8\pi.$$

Although it seemed like everything worked out nicely, the above answer of 8π is wrong. The problem is that the function $\sin(3\theta)$ is not nonnegative on the interval $0 \leq \theta \leq 2\pi$.

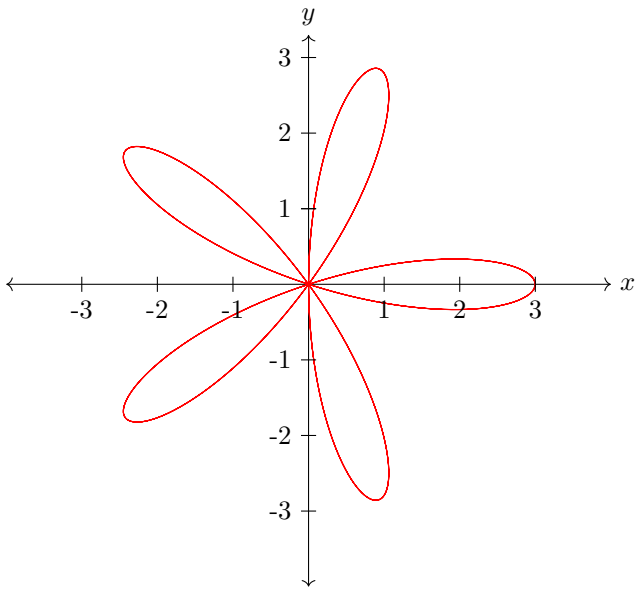
The correct area is 4π, obtained by observing that there are 3 petals and the area of a single petal is $\frac{4}{3}\pi$ by Example 41.

End of Example 45. □

Example 46. Find the area enclosed by one loop of the curve

$$r = 3\cos(5\pi).$$

The plot of this is shown below:



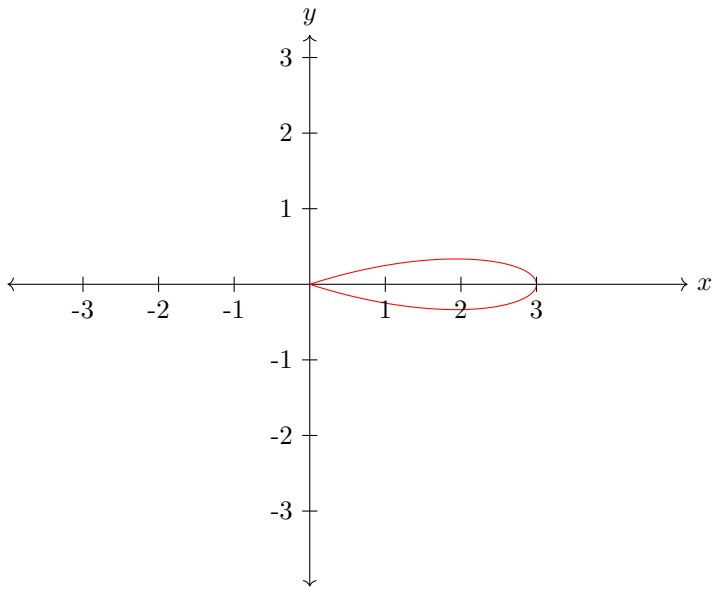
What should we pick as the interval? One way to figure this out is to look at when the polar function crosses the origin, i.e., when $r = 0$:

$$\begin{gathered} r = 0 \\ \Updownarrow \\ \cos(5\theta) = 0 \\ \Updownarrow \\ 5\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots \\ \Updownarrow \\ \theta = \pm\frac{\pi}{10}, \pm\frac{3\pi}{10}, \pm\frac{5\pi}{10}, \dots \end{gathered}$$

Here, the smallest angles about $\theta = 0$ are $\theta = \pm\frac{\pi}{10}$. So the equations

$$\begin{cases} r = 3\cos(5\pi) \\ -\frac{\pi}{10} \leq \theta \leq \frac{\pi}{10} \end{cases}$$

traces out the following petal:



The area of enclosed by this loop is

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/10}^{\pi/10} r^2 d\theta \\
&= \frac{1}{2}\int_{-\pi/10}^{\pi/10} \left[3\cos(5\theta)\right]^2 d\theta \\
&= \frac{9}{10}\int_{-\pi/2}^{\pi/2} \cos^2(\theta) d\theta && \text{subtitution: } u = 5\theta \\
&= \frac{9}{10}\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_{-\pi/2}^{\pi/2} && \text{by Proposition 44} \\
&= \frac{9}{10}\left[\frac{\pi}{2}\right] \\
&= \frac{9\pi}{20}.
\end{aligned}$$

End of Example 46. □

12 2026-09-21 | Week 05 | Lecture 12

Topic of this lecture: *3-dimensional space*
Reading assignment: *chapter 12.1, start 12.2. Watch this video:* `https://www.3blue1brown.com/lessons/vectors`

12.1 3D Space

Based on sections 12.1 in the texbook

3D space is denoted by the symbol $\mathbb{R}^3$. A point in 3D space is specified as an ordered triple

$$(x, y, z)$$

where $x, y, z \in \mathbb{R}$. Here, z is the distance above (or below) the xy-plane.

Many familiar equations take on new meaning in 3D space, compared to 2D space.

- Consider the equation
$$\begin{cases} x^2 + y^2 = 1 \\ z = 3 \end{cases}$$
This is a unit circle floating 3 units above the xy-plane.
On the other hand, the equation
$$x^2 + y^2 = 1$$
imposes no restriction on the value of z. Consequently, this is the equation of a (hollow) cyclinder.
- The equation
$$x^2 + y^2 + z^2 = 1$$
is the equation of a sphere with radius 1 centered at the origin.

12.2 Introduction to vectors

Based on textbook section 12.2

What is a vector? Here are three definitions:

- **Algebraic definition:** A ***vector*** is a list of numbers.
- **Geometric definition:** the difference between two points in space.
- **Physical definition:** an arrow with direction and magnitude.

[picture]

Example 47 (Vector)**.** Consider three points in the plane

$$A = (4, 3), \quad \text{and} \quad B = (1, 2)$$

Then, for example, we have the vector $\overrightarrow{AB} = \langle 4 - 1, 3 - 2\rangle = \langle 3, 1\rangle$.

Notation 48. The textbook uses the angled brackets notation $\langle x, y\rangle$ for vectors. We will sometimes use

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or

$$(x, y) \quad \text{or} \quad (x, y, z)$$

All of these notations are common. The entries of a vector (e.g., x, y, etc) are called ***components***.

The book denotes vectors with **bold** letters, like $\mathbf{v}$ or $\mathbf{x}$. Sometimes people write $\vec{v}$ or $\vec{x}$, which is easier to write by hand.

End of Example 48. □

Example 49 (Vector addition and scaling)**.** Let $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle -3, -1\rangle$. Find the sum of the fetors and illustrate geometrically. Then

$$\mathbf{u} + \mathbf{v} = \langle -2, 1\rangle .$$

and you can add in any order

$$\mathbf{v} + \mathbf{u} = \langle -2, 1\rangle .$$

What about $\langle -1, 4\rangle$ and $\langle 6, -2\rangle$?

We can have 3d vectors, like

$$\langle 3, -2, 5\rangle$$

and multiplying a vector by a number a (called a ***scalar***) *scales* the vector by stretching it out or whatever. For example, compare the vectors the vector

$$\mathbf{u} = \langle 2, 3\rangle \quad \text{and} \quad 2 \cdot \mathbf{u} = \langle 4, 6\rangle$$

End of Example 49. □

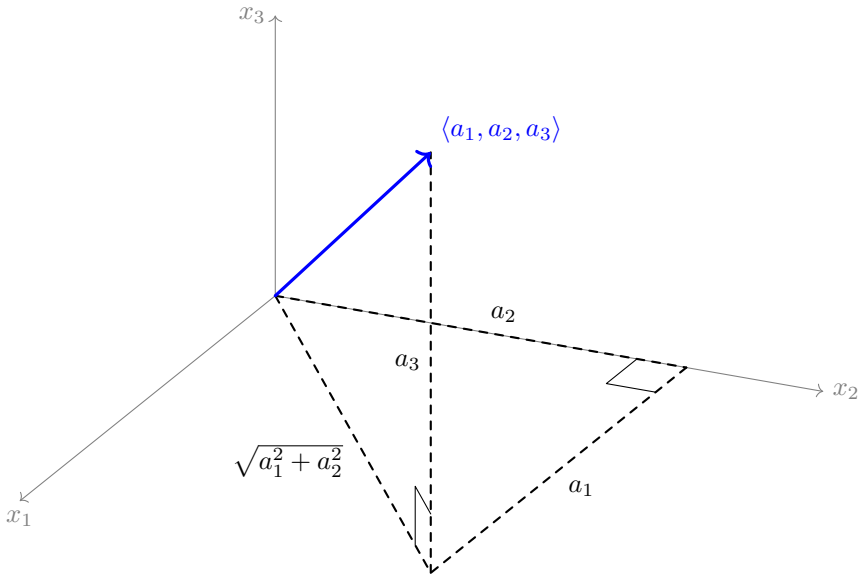
Proposition 50. *The length of a 2D vector* $\mathbf{a} = (a_1, a_2)$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

and the length of a 3D vector $\mathbf{a} = (a_1, a_2, a_3)$ *is*

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

This is shown below



The ***zero vector***, denoted $\mathbf{0}$, is the vector whose components are all equal to zero. A vector of length 1 is called a ***unit vector***. If $\mathbf{v} \neq \mathbf{0}$, then you can ***normalize*** $\mathbf{v}$ by dividing it by its length. That is, let

$$\mathbf{u} := \frac{\mathbf{v}}{|\mathbf{v}|}.$$

When you do this, the vector $\mathbf{u}$ is a unit vector.

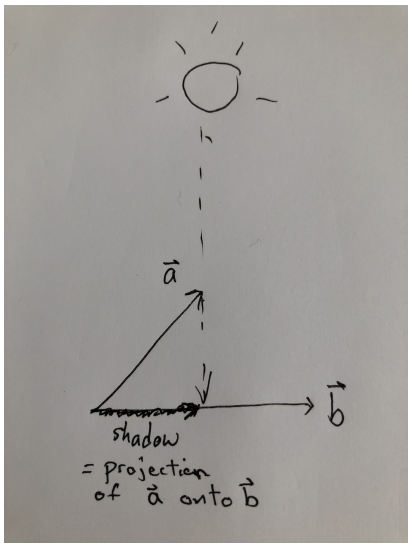
13 2026-09-23 | Week 05 | Lecture 13

Topic of this lecture: *vector algebra, dot product*
Reading assignment: *chapter 12.1 through 12.3*

13.1 Eratosthenes' estimation of the Earth's circumference

To motivate the present discussion of vectors, I presented Eratosthenes' famous calculation in which he correctly estimated the circumference of the Earth over 2200 years ago using nothing but sticks, shadows, and trigonometry.

We are interested in understanding—in a precise, mathematical way—the *shadow* of a vector as it falls onto another vector, as shown:



Eratosthenes' calculation (which isn't written up in these notes) suggests that understanding shadows like this can be a powerful mathematical tool.

The shadow above is an example of a *vector projection*. To understand vector projections, we'll need to have a firm grasp of the following things

- lengths of vectors
- the geometry of vectors and angles between them
- how to scale and rescale vectors
- and some more things that aren't apparent yet.

13.2 The standard basis vectors

Definition 51 (Standard Basis Vectors)**.** In 3 dimensions, the ***standard basis vectors*** are

$$\mathbf{i} = e_1 = \langle 1, 0, 0 \rangle \tag{16}$$
$$\mathbf{j} = e_2 = \langle 0, 1, 0 \rangle \tag{17}$$
$$\mathbf{k} = e_3 = \langle 0, 0, 1 \rangle \tag{18}$$

Sometimes we use the notation $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and other times we use the notation e_1, e_2, e_3.

Any vector in 3D can be written as a sum of the standard basis vectors, like this:

$$\begin{aligned}\langle 5, -4, 2\rangle &= 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \\ &= 5e_1 - 4e_2 + 2e_3\end{aligned}$$

Example 52. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$. Express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$:

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= (2 + 12)\mathbf{i} + 4\mathbf{j} + (-6 + 21)\mathbf{k} \\ &= 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}.\end{aligned}$$

End of Example 52. □

The vector with length 0 is called the ***zero vector***. It's the vector

$$\mathbf{0} = \langle 0, 0\rangle \quad \text{or} \quad \mathbf{0} = \langle 0, 0, 0\rangle$$

depending on what space you are working with.

13.3 Scaling vectors

Example 53 (Scaling a vector)**.** In general, one can ***scale*** a vector by multiplying it by a number. Let

$$\mathbf{v} = \langle -2, 4, 4\rangle$$

Then $\mathbf{v}$ has length

$$|\mathbf{v}| = \sqrt{(-2)^2 + (4)^2 + (4)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6.$$

If we scale $\mathbf{v}$ by a factor of 3, then the resulting vector is

$$\mathbf{u} = 3\mathbf{v} = \langle -6, 12, 12\rangle$$

which has three times the length:

$$|\mathbf{u}| = \sqrt{(-6)^2 + (12)^2 + (12)^2} = \sqrt{36 + 144 + 144} = \sqrt{324} = 18.$$

Similarly, if I scale a vector by 2, then its length gets doubled. If I scale a vector by $1/2$, its length gets halved.

End of Example 53. □

13.4 Distance between points

Example 54 (Distance between points)**.** The formula for computing the distance between two points in 3D space is related to the formula for computing the length of a 3D vector. Say you have a fly which starts at point $P := (x_1, y_1, z_1)$, flies around around a bit, and then terminates at point $Q := (x_2, y_2, z_2)$. Then the ***displacement vector*** is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

You can think of this as an arrow drawn from the fly's starting position P to its ending position Q. The distance between points P and Q is nothing other than the magnitude of this vector. Therefore, distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\text{distance } = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

For example, say the fly starts at $(3, 2, 9)$ and ends at $(2, -2, 1)$. The distance between these two points is

$$\begin{aligned}\text{distance} &= \sqrt{(3-2)^2 + (2-(-2))^2 + (9-1)^2} \\ &= \sqrt{1^2 + 4^2 + 8^2} \\ &= \sqrt{1 + 16 + 64} \\ &= \sqrt{81} \\ &= 9.\end{aligned}$$

End of Example 54. □

13.5 Dot Product

Definition 55 (Dot Product)**.** Let $\mathbf{a}$ and $\mathbf{b}$ be vectors. The ***dot product*** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

A similar definition holds for vectors of length 2 (or 10, or 35, etc).

Example 56. Let $\mathbf{a} = \langle 1, -4, 1 \rangle$, $\mathbf{b} = \langle 0, 2, -2 \rangle$, and $\mathbf{c} = \langle 5, 2, 1 \rangle$ Then

$$\mathbf{a} \cdot \mathbf{b} = (1)(0) + (-4)(2) + (1)(-2) = 0 - 8 - 2 = -10.$$

Similarly,

$$\mathbf{a} \cdot \mathbf{c} = 5 - 8 + 1 = -2 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{c} = 0 + 4 - 2 = 2.$$

End of Example 56. □

The dot product has important geometric meaning, which we will explore. But first, let's make some observations about the properties of the dot product, summarized in the following proposition:

Proposition 57 (Dot Product Properties)**.** *Let* $\mathbf{a} = (a_1, a_2, a_3)$*, and* $\mathbf{b} = (b_1, b_2, b_3)$ *and* $\mathbf{c} = (c_1, c_2, c_3)$ *be vectors. Then the following properties hold*

- $\mathbf{0} \cdot \mathbf{a} = 0$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

The next theorem connects the dot product of two vectors with the angle between them.

Theorem 58 (The dot-product/angle relationship)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors. Suppose that the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is* θ*. Then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Let's try to unwrap some implications of this theorem:

- When θ is an acute angle, that means $0 < \theta < \frac{\pi}{2}$. In that case, $\cos(\theta) > 0$, so $\mathbf{a} \cdot \mathbf{b} > 0$.
- On the other hand, when θ is an obtuse angle, that means $\frac{\pi}{2} < \theta < \pi$. In that case, $\cos(\theta) < 0$, and so $\mathbf{a} \cdot \mathbf{b} < 0$.
- When we have a right angle, $\theta = \frac{\pi}{2}$, so in that case $\mathbf{a} \cdot \mathbf{b} = 0$.

Theorem 59 (The dot-product/angle relationship)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors. Suppose that the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is* θ*. Then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Proposition 60 (Orthogonality condition)**.** *We say that two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are* ***orthogonal*** *(or equivalently,* ***perpendicular****) if the angle between them is* $\theta = \frac{\pi}{2}$*. Theorem 59 implies* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal precisely when*

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Morover, whether the dot product is positive or negative tells us whether the angle θ *is acute or obtuse.*

A direct consequence of Theorem 59 and the fact that $|\cos(\theta)| \leq 1$ is the following theorem:

Theorem 61 (Cauchy-Schwarz Inequality)**.**

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

14 2026-09-25 | Week 05 | Lecture 14

Topic of this lecture: *vector projections*
Reading assignment: *chapter 12.3*

14.1 Warm-up

Example 62. Find the angles between the following vectors

- $\mathbf{a} = \langle 2, 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 3, 1 \rangle$
- $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -4, 2 \rangle$ (perpendicular)

End of Example 62. □

Example 63. Consider the triangle consisting of the following 3 points:

- $P = (1, -3, -2)$
- $Q = (2, 0, -4)$
- $R = (6, -2, -5)$

Is this triangle a right triangle? Since the triangle lives in 3D spaces, so it's not so easy to simply plot it and look. Use vectors to determine if it is a right triangle.

End of Example 63. □

14.2 Projection

Definition 64 (Vector Projection)**.** Let $\mathbf{a}, \mathbf{b}$ be vectors. The ***vector projection of*** $\mathbf{b}$ ***onto*** $\mathbf{a}$ is

$$\begin{aligned}\mathbf{P} &:= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\left(\frac{\mathbf{a}}{|\mathbf{a}|}\right) \\ &= (\text{signed length of } P)\,(\text{direction of } \mathbf{a})\end{aligned}$$

The book uses the notation $\mathbf{P} = \text{proj}_{\mathbf{a}}(\mathbf{b})$.

15 2026-09-28 | Week 06 | Lecture 15

Topic of this lecture: *vector projections*
Reading assignment: *chapter 12.3*

15.1 Projection

Definition 65 (Vector Projection)**.** Let $\mathbf{a}, \mathbf{b}$ be vectors. The ***vector projection of*** $\mathbf{b}$ ***onto*** $\mathbf{a}$ is

$$\begin{aligned}\mathbf{P} &:= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\left(\frac{\mathbf{a}}{|\mathbf{a}|}\right)\\ &= (\text{signed length of } P)\,(\text{direction of } \mathbf{a})\end{aligned}$$

The book uses the notation $\mathbf{P} = \text{proj}_{\mathbf{a}}(\mathbf{b})$.

To break this equation down, observe that $\frac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector in the direction of $\mathbf{a}$. On the other hand, the quantity

$$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)$$

is a number, not a vector. It scales the unit vector $\frac{\mathbf{a}}{|\mathbf{a}|}$. This number is the (signed) length of the shadow cast by $\mathbf{b}$ on $\mathbf{a}$ when the sun is directly overhead.

Example 66. A plane is flying in the direction $\mathbf{D} = \langle 1, 1\rangle$. The plane is getting hit by a tailwind with wind velocity $\mathbf{V} = \langle 100, 0\rangle$. Find $\mathbf{P}$, the projection of $\mathbf{V}$ onto $\mathbf{D}$. Here, $\mathbf{V}$ points east and $\mathbf{D}$ points northeast.

Applying the formula from Definition 65, we want to compute the following:

$$\mathbf{P} = \left(\frac{\mathbf{D}\cdot\mathbf{V}}{|\mathbf{D}|}\right)\left(\frac{\mathbf{D}}{|\mathbf{D}|}\right).$$

We first compute $\mathbf{D}\cdot\mathbf{V}$ and $|\mathbf{D}|$:

$$\mathbf{D}\cdot\mathbf{V} = 100, \qquad |\mathbf{D}| = \sqrt{2}.$$

Hence

$$\mathbf{P} = \left(\frac{100}{\sqrt{2}}\right)\left(\frac{\mathbf{D}}{\sqrt{2}}\right) = 50\,\langle 1, 1\rangle = \langle 50, 50\rangle\,.$$

End of Example 66. □

Example 67. Find the vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.
Solution: We first compute $\mathbf{a}\cdot\mathbf{b}$ and $|\mathbf{a}|$:

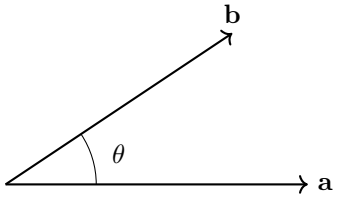
$$\mathbf{a}\cdot\mathbf{b} = -2 + 3 + 2 = 3, \qquad |\mathbf{a}| = \sqrt{4 + 9 + 1} = \sqrt{14}.$$

Therefore

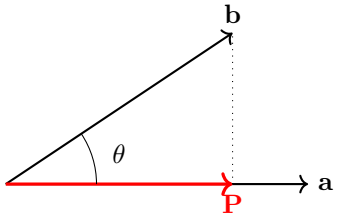
$$\begin{aligned}P &= \left(\frac{3}{\sqrt{14}}\right)\frac{\mathbf{a}}{\sqrt{14}}\\ &= \frac{3}{14}\,\langle -2, 3, 1\rangle\\ &= \left\langle -\frac{6}{14}, \frac{9}{14}, \frac{3}{14}\right\rangle.\end{aligned}$$

End of Example 67. □

Question: Why is $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$ the signed length of $\mathbf{P}$?
Answer: Suppose we've got two vectors like this:



Now let's draw the projection $\mathbf{P}$, along with a dotted line perpendicular to $\mathbf{a}$:



Note that the above depicts a right angle. Since $\cos = \frac{\text{adj}}{\text{hyp}}$,

$$\cos(\theta) = \frac{\text{length of } \mathbf{P}}{\text{length of } \mathbf{b}} = \frac{|\mathbf{P}|}{|\mathbf{b}|} \tag{19}$$

On the other hand, by Theorem 59, we also know that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}. \tag{20}$$

Therefore, combining Eqs. (19) and (20), we have:

$$\frac{|\mathbf{P}|}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

which simplifies to

$$\boxed{|\mathbf{P}| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}}$$

This answers our original question. [To be precise, here we have actually only considered the case where θ is an acute angle. The case where θ is an obtuse angle is a little bit different, but not much.]

16 2026-09-30 | Week 06 | Lecture 16

Topic of this lecture: *force*
Reading assignment: *12.3, 12.4*

16.1 Connection to Physics: Force

Vectors are used to represent ***force***, understood in physics as a physical interaction (e.g. a push or a pull) which causes (or can cause) change in an object's motion, shape, or resting position. Force has magnitude (i.e., length) and a direction.

Example 68 (Forces)**.** If you are floating in the ocean, you get pushed around by lots of forces: (1) gravity, (2) waves, (3) current, (4) wind, (5) forces created by your own motion, (6) buoyant force, and others. Each of these forces can be modeled by a vector. The magnitude of the vector indicates the "strength" of the corresponding force. For example, wind doesn't exert much force on a swimmer, so that vector might be tiny. By comparison the vector for current might be very large.

The ***resultant force*** is the sum of these vectors, and determines how your velocity changes.

End of Example 68. □

Force is measured in ***Newtons***, denoted $\mathtt{N}$. How much is one $1\mathtt{N}$? It's approximately equal to the weight that you feel pushing down on your hand when you hold up a stick of butter. (Note: *weight* is a force).

Technically, $\mathtt{N} = \mathtt{kg} \cdot \mathtt{m}/\mathtt{s}^2$. In words one Newton is the amount of force needed to accelerate one kg of mass at one meter per second per second (e.g., so that after two seconds the mass is moving at two meters per second). This is a bit confusing at first, as it is the product of a unit of mass ($\mathtt{kg}$) with a unit of acceleration ($\mathtt{m}/\mathtt{s}^2$). This is due to ***Newton's Second Law***, which states

$$\text{force} = \text{mass} \times \text{acceleration}.$$

or

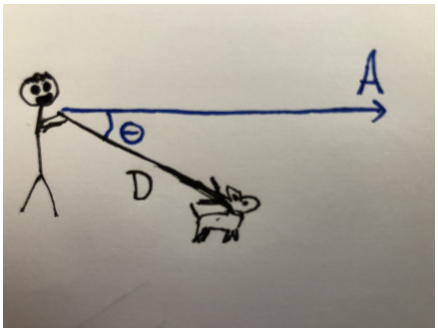
$$\mathbf{F} = m\mathbf{a}$$

Here both $\mathbf{F}$ and $\mathbf{a}$ are vectors, m is a scalar.

16.2 Connection to physics: Effective force in a direction

If a force $\mathbf{F}$ is applied to a particle moving along a path, we may wish to know the magnitude of the force in the direction of motion. If $\mathbf{A}$ is a direction, then the projection of $\mathbf{F}$ along $\mathbf{A}$ is the ***effective force*** in the direction $\mathbf{A}$.

Example 69. Suppose you are walking a dog on a walk and that you are facing in direction $\mathbf{A}$. Suppose the dog lunges in some other direction. Let $\mathbf{D}$ be force exerted by the pull of the leash. Then the effective force with which the dog pulls you forward is the projection $\mathbf{P}$ of $\mathbf{D}$ along $\mathbf{A}$.



If $\mathbf{A} = \{1, 1, 0\}$ and $\mathbf{D} = \langle 13, 12, -5\rangle$, then

$$\begin{aligned}\mathbf{P} &= \left(\frac{\mathbf{A}\cdot\mathbf{D}}{|\mathbf{A}|}\right)\left(\frac{\mathbf{A}}{|\mathbf{A}|}\right)\\ &= \left(\frac{13+12+0}{\sqrt{2}}\right)\left(\frac{\langle 1,1,0\rangle}{\sqrt{2}}\right)\\ &= \left(\frac{25}{\sqrt{2}}\right)\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\rangle\\ &= \left\langle \frac{25}{2}, \frac{25}{2}, 0\right\rangle.\end{aligned}$$

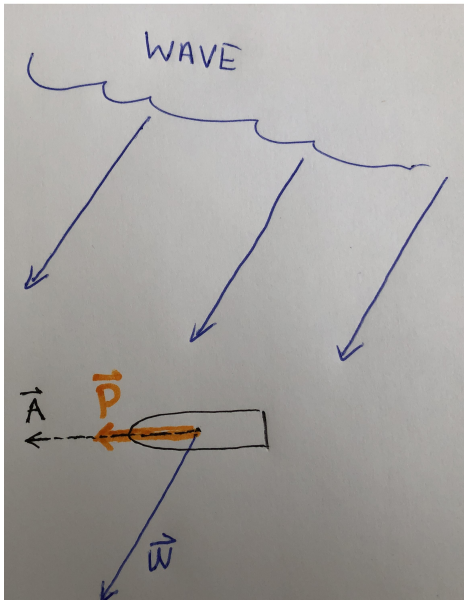
The length of $\mathbf{P}$ is

$$|\mathbf{P}| = \frac{\mathbf{A}\cdot\mathbf{D}}{|\mathbf{A}|} = \frac{25}{\sqrt{2}} \approx 17.7.$$

So the dog pulls you forward with about 17.7 N of force. That's not enough to topple you unless you are either very frail, or a toddler.

End of Example 69. □

Example 70. Suppose we have a surfer pointed in direction $\mathbf{A}$, who catches a wave which exerts a force (represented by the vector $\mathbf{W}$) on the surfboard, as shown:



Let's assume that $\mathbf{A} = \langle -4, -1\rangle$ and that $\mathbf{W} = \langle -700, -600\rangle$. The length of vector $\mathbf{A}$ doesn't matter; this vector is just here to indicate the direction of the surfboard. We are assuming that the surfboard starts out stationary. On the other hand, -700 and -600 are measured in Newtons. (These vectors aren't quite to scale in the above picture). The effective force of the wave moving surfboard forward in the $\mathbf{A}$ direction is the projection of $\mathbf{W}$ along $\mathbf{A}$, which we will denote by $\mathbf{P}$.

To compute this, we will first do some preliminary computations:

$$|\mathbf{A}| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17},$$

and

$$\mathbf{A}\cdot\mathbf{W} = \langle -4, -1\rangle \cdot \langle -700, -600\rangle = 2800 + 600 = 3400.$$

Then

$$\begin{aligned}\mathbf{P} &= \left(\frac{\mathbf{A}\cdot\mathbf{W}}{|\mathbf{A}|}\right)\left(\frac{\mathbf{A}}{|\mathbf{A}|}\right) && \text{by Definition 65.}\\ &= \left(\frac{3400}{\sqrt{17}}\right)\left(\frac{\langle -4,-1\rangle}{\sqrt{17}}\right)\\ &= \frac{3400}{17}\langle -4,-1\rangle\\ &= \langle -800,-200\rangle\,.\end{aligned}$$

Here, the initial force of the wave was $\mathbf{W}$, which had a magnitude of

$$|\mathbf{W}| = \sqrt{(-700)^2+(-600)^2} \approx 920\ \mathtt{N}$$

The "effective force in the direction of motion" is $\mathbf{P} = \langle -800,-200\rangle$, which has a slightly smaller magnitude than the inital force:

$$|\mathbf{P}| = 100\sqrt{64+4} \approx 820\mathtt{N}$$

Question: How much does the surfboard accelerate in direction $\mathbf{A}$? Assume the surfer weighs $70\mathtt{kg}$.

By Newton's second law,

$$\mathbf{F} = m\mathbf{a}$$

and therefore,

$$\begin{aligned}\mathbf{a} &= \frac{\mathbf{F}}{m}\\ &= \frac{820\ \mathtt{N}}{70\ \mathtt{kg}}\end{aligned}$$

Recall that $\mathtt{N} = \mathtt{kg}\times\frac{\mathtt{m}}{\mathtt{s}^2}$, so $\frac{\mathtt{N}}{\mathtt{kg}} = \frac{\mathtt{m}}{\mathtt{s}^2}$

$$a = \frac{820}{70}\frac{\mathtt{m}}{\mathtt{s}^2} \approx 11.7\mathtt{m}/\mathtt{s}^2$$

This means that if the force lasts for half a second, the surfer will be accelerated to about $\frac{11.7}{2} = 5.85$ meters per second in the $\mathbf{A}$-direction.

End of Example 70. □

17 2026-10-02 | Week 06 | Lecture 17

Question: How do we describe (and find) the tilt of a plane?
Reading assignment: *Chapter 12.4*

17.1 Cross Product

The idea of "slope" allows us to describe the tilt of a line. We need something analogous for describing the tilt of a plane. This is done with by describing a vector that is *perpendicular* to the plane, called the ***normal vector***. So we need a way to find vectors that are perpendicular to things. In 3D, the simplest way to do this is with the following operation (it's not really simple, sorry):

Definition 71 (Cross Product)**.** Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The ***cross product*** of $\mathbf{a}$ and $\mathbf{b}$, denoted $\mathbf{a} \times \mathbf{b}$, is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Fact 1: This new vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
Fact 2: There is no cross product for 2D.

There is a trick to computing cross products, and remembering how to compute them. This is called the ***method of cofactor expansion***, which I will demonstrate now. This method touches on important ideas that arise in linear algebra, which gives it additional value in its own right. Given a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant of the matrix, which we denote by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is the quantity

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Let

$$\mathbf{i} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{j} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the cross product $\mathbf{a} \times \mathbf{b}$ can be computed in the following way:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Note that the sign *alternates:* the term with the $\mathbf{j}$ is subtracted, not added like the first and third term.

Example 72. Suppose $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$. Then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} &= \mathbf{i} \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \\ &= \mathbf{i}\,(-15 - 28) - \mathbf{j}\,(-5 - 8) + \mathbf{k}\,(7 - 6) \\ &= \mathbf{i}\,(-43) - \mathbf{j}\,(-13) + \mathbf{k}\,(1) \\ &= \langle -43, 13, 1 \rangle \end{aligned}$$

Check our answer: Is this new vector $\mathbf{a} \times \mathbf{b}$ indeed orthogonal to both $\mathbf{a}$ and $\mathbf{b}$? To check this, we can use the fact that two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal exactly when $\mathbf{u} \cdot \mathbf{v} = 0$. Indeed,

$$\mathbf{a} \cdot \langle -43, 13, 1 \rangle = \langle 1, 3, 4 \rangle \cdot \langle -43, 13, 1 \rangle = -43 + 39 + 4 = 0$$

Since this dot product is zero, we conclude that our new vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$. Similarly, we have

$$\mathbf{b} \cdot \langle -43, 13, 1 \rangle = \langle 2, 7, -5 \rangle \cdot \langle -43, 13, 1 \rangle = -86 + 91 - 5 = 0,$$

and this implies that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

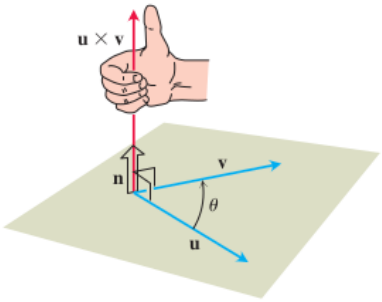
End of Example 72. □

The next theorem establishes a geometric interpretation of the cross product:

Theorem 73. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then the vectors determine a plane. The plane is*

$$P = \left\{ (x, y, z) \in \mathbb{R}^3 : (x, y, z) = c_1\mathbf{u} + c_2\mathbf{v} \text{ for some } c_1, c_2 \in R^2 \right\}$$

An expression of the form $c_1\mathbf{u} + c_2\mathbf{v}$ is called a ***linear combination*** *of $\mathbf{u}$ and $\mathbf{v}$. Thus, the plane P is set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$. We select a unit vector $\mathbf{n}$ called the* ***unit normal vector*** *according to the right hand rule as follows.*



In words, if your fingers curl from $\mathbf{u}$ to $\mathbf{v}$, then $\mathbf{n}$ points in the direction of your thumb. Then

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin(\theta))\,\mathbf{n}. \tag{21}$$

where $\theta \in [0, \pi]$ is the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$.

This geometric interpretation comes with a number of implications:

(i.) Eq. (21) implies that two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel (i.e., so that $\theta = 0$), precisely when

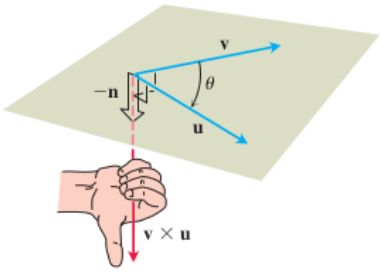
$$\mathbf{u} \times \mathbf{v} = 0.$$

(ii.) It verifies the fact, stated earlier, that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(iii.) The cross product is *anticommutative*:

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$$

Because when computing $\mathbf{v} \times \mathbf{u}$, your fingers have to sweep from $\mathbf{v}$ to $\mathbf{u}$, which causes your thumb to point the other way, as shown:



And geometrically, reversing the direction of vector is the same as mupltiplying it by -1. This is is why

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$$

18 2026-10-04 | Week 07 | Lecture 18

Topic of this lecture: *cross product, torque*
Reading assignment: *chapter 12.4*

18.1 Cross product: geometric interpretation

Recall we had the following theorem:

Theorem 74. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ *and let* $\theta \in [0, \pi]$ *be the angle between them. Then*

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin(\theta))\mathbf{n} \tag{22}$$

Here are two properties that follow from Eq. (22):

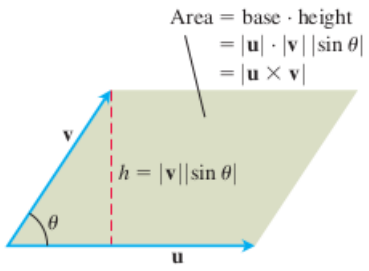
(i) If $a, b \in \mathbb{R}$ then

$$(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$$

(ii) We get a nice way to compute the *magnitude* of the cross product:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta),$$

and we get a neat interpretation too: the magnitude of the cross product equals the area of the parallelogram:



18.2 Application to physics: torque

Suppose you have a wrench, whose handle is represented by a vector $\mathbf{r}$. The wrench is attached to a bolt. Pushing on the wrench causes it to turn around an axis of rotation (the bolt). The turning effect is described by a vector called the ***torque vector***, denoted $\boldsymbol{\tau}$, which is similar to—but *not* the same as—a force vector.

The magnitude of $\boldsymbol{\tau}$ determines how much push there is to rotate:

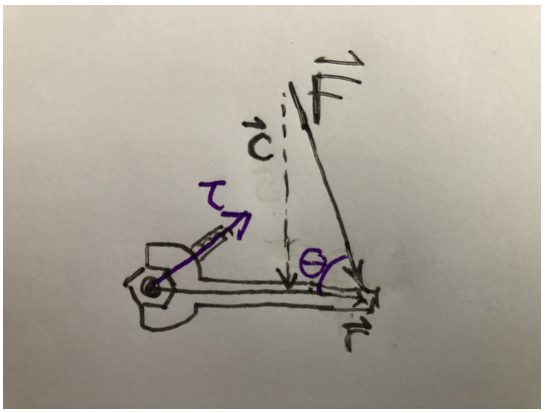
$$\text{bigger magnitude} = \text{more push to rotate}$$

This is similar to a force vector. What is different is that for a force vector $\mathbf{F}$, the direction of $\mathbf{F}$ indicates the direction in which motion would occur. This is NOT the case for torque.

Instead, the direction of $\boldsymbol{\tau}$ points in a direction *parallel to the bolt*. Whether it points up or down is determined by whether the rotation is clockwise or counter-clockwise. For a standard bolt (which tightens when turned clockwise), the direction is easy to remember:

- turning clockwise tightens the bolt, and the torque arrow points down;
- turning counter-clockwise loosens the bolt, and the torque arrow points up.

Suppose that the bolt is stuck, so I take a hammer and hit the end of my wrench. The hammer strike is represented by a force vector $\mathbf{F}$, as shown:



Notice that I didn't have great aim, so although my hammer hit the tip of the wrench, the hammer didn't come down exactly perpendicular to the wrench. Instead, it was at a bit of an angle θ.

In this setting, if we think about this physically, we will conclude that the magnitude of the torque should be determined by two things:

- The length of the wrench handle $|\mathbf{r}|$.
- How much of the force gets applied in the direction perpendicular to $\mathbf{r}$. In the picture above, this is the magnitude of the vector $\mathbf{c}$. We can compute $|\mathbf{c}|$ by observing that

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{|\mathbf{c}|}{|\mathbf{F}|},$$

so that

$$|\mathbf{c}| = |\mathbf{F}|\sin(\theta).$$

It turns out that the cross product is the right thing to use to represent the torque. That is, the precise definition of ***torque*** is

$$\boldsymbol{\tau} := \mathbf{r} \times \mathbf{F}.$$

Using Eq. (21), the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = \underbrace{|\mathbf{r}|}_{\substack{\text{length of} \\ \text{wrench}}} \underbrace{|F|\sin(\theta)}_{|\mathbf{c}|}.$$

Example 75 (Torque)**.** Suppose we are attempting to turn a bolt using a wrench of length $\mathbf{r} = .25$ `m` and we apply a force $\mathbf{F}$ of 40 `N` at an angle of 75°. What is the magnitude of the torque applied to the bolt?

$$\begin{aligned} |\boldsymbol{\tau}| &= |\mathbf{r} \times \mathbf{F}| \\ &= |\mathbf{r}||\mathbf{F}|\sin(\theta) \\ &= (0.25\ \texttt{m})(40\ \texttt{N})\sin(75^\circ) \\ &\approx 9.66\ \texttt{Nm} \\ &= 9.66\ \texttt{J} \end{aligned}$$

where we used the fact that `1 Joule = 1 Netwon-meter`.

The torque vector is

$$\boldsymbol{\tau} \approx 9.66\mathbf{n}$$

What direction does $\mathbf{n}$ point? If we are tightening the bolt then the rotation is clockwise, and the torque vector points away from us. (This matches with the righty-tighty lefty-loosy convention that almost all bolts and screws obey.)

End of Example 75. □

19 2026-10-07 | Week 07 | Lecture 19

Topic of this lecture: *lines and planes in 3-dimension*
Reading assignment: *chapter 12.5*

19.1 Lines in $\mathbb{R}^3$

Definition 76 (***The vector equation of a line***)**.** Let $\mathbf{p} \in \mathbb{R}^3$ be a point and let $\boldsymbol{v} \in \mathbb{R}^3$ be a nonzero vector. Consider the line L parallel to $\boldsymbol{v}$ which passes through $\boldsymbol{p}$. We may express L as

$$\boldsymbol{r}(t) = \mathbf{v}t + \mathbf{p}, \qquad -\infty < t < \infty. \tag{23}$$

This is called a ***vector equation*** for L.

Eq. (23) can be written as

$$\boldsymbol{r}(t) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} t + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \qquad -\infty < t < \infty.$$

This may be expressed as the ***parametric equations for the line L:***

$$\begin{cases} x = v_1 t + p_1 \\ y = v_2 t + p_2 \\ z = v_3 t + p_3 \\ -\infty < t < \infty, \end{cases}$$

which are called the ***parametric equations of a line.***

Example 77. Suppose L is the line in $\mathbb{R}^2$ through the point $\mathbf{p} = (1, 2, 3)$ in the direction of $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$.

Then the vector equation of the line is

$$\mathbf{r}(t) = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad -\infty < t < \infty.$$

The parametric equations are

$$\begin{cases} x = t + 1 \\ y = -3t + 2 \\ z = 4t + 3 \end{cases} \qquad -\infty < t < \infty.$$

If we eliminate t, we get the equation

$$y = -3x + 5.$$

End of Example 77. □

Example 78. A dragonfly flies from a lilypad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 40 ft per second. What is the position of the dragonfly after 10 seconds?

Solution: The normalized direction is

$$\boldsymbol{u} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

The dragonfly's position at time t is

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}t,$$

where $\boldsymbol{r}(0) = (0,0,0)$ is its position at time $t = 0$, and $\boldsymbol{v}$ is its velocity. We may write the velocity as

$$\begin{aligned}\boldsymbol{v} &= (\text{speed})(\text{direction}) \\ &= |\boldsymbol{v}|\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right) \\ &= (40\texttt{ft/sec}) \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.\end{aligned}$$

Therefore

$$\boldsymbol{r}(10) = \begin{bmatrix} 400/\sqrt{3} \\ 400/\sqrt{3} \\ 400/\sqrt{3} \end{bmatrix} \texttt{ feet}$$

End of Example 78. □

Example 79. Consider the following two points in 3D space

$$A = (2,4,-3) \quad \text{and} \quad B = (3,-1,1)$$

(a) Find the equation of the line which passes through A and B

(b) Find the equation of the line segment which connects A and B.

Solution to (a): We will follow the ideas presented in Definition 76. Let $\mathbf{v} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$. Let $\mathbf{p} = A$. (We could take $\boldsymbol{p} = B$ instead, if we want). Then the vector equation of the line is

$$\boldsymbol{r}(t) = t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}, \qquad -\infty < t < \infty$$

and the parameteric equations of the line are

$$\begin{cases} x = t + 2 \\ y = -5t + 4 \qquad -\infty < t < \infty \\ z = 4t - 3 \end{cases}$$

Solution to (b): Since $\overrightarrow{AB} = (B - A)$, observe that our equation is of the form

$$\boldsymbol{r}(t) = \overrightarrow{AB}t + A$$

or

$$\boldsymbol{r}(t) = (B - A)t + A$$

Hence

$$\boldsymbol{r}(0) = A, \qquad \boldsymbol{r}(1) = B$$

Therefore the equation of the line segement between A and B is

$$\boldsymbol{r}(t) = t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}, \qquad 0 \le t \le 1$$

End of Example 79. □

19.2 Planes in $\mathbb{R}^3$

Recall that, geometrically, a line is determined by two features:

1. A slope m which determines the direction of the line
2. A point (x_0, y_0) which the line passes through, as this determines where the line lives on the xy-plane

It is easy to see that these two things determine everything about a line because the equation of a line can be expressed as

$$y - y_0 = m(x - x_0)$$

and to write this down, all we need are m and (x_0, y_0).

Just like a line, a plane is determined by two things:

1. A normal vector $\boldsymbol{n} = \langle A, B, C\rangle$ which determines the tilt of the plane. (Here, $A, B, C \in \mathbb{R}$.)
2. A point $(x_0, y_0, z_0) \in \mathbb{R}^3$ which the plane passes through, to determine the location in 3D space the plane sits.

To be precise, in 3-dimensional space $\mathbb{R}^3$, a plane $\mathbb{P}$ consists of the set of points (x, y, z) satisfying the following equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{24}$$

This is called the ***standard form*** (or ***normal form***) equation of a plane, and we can write it down if we know both $\boldsymbol{n} = \langle A, B, C\rangle$ and (x_0, y_0, z_0). So if we know those two things, then we know the equation of the plane, meaning we know everything about it.

By a little bit of algebra, we can rewrite Eq. (24) as

$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$.

We can also write Eq. (24) as the following dot product:

$$\underbrace{\begin{bmatrix} A \\ B \\ C \end{bmatrix}}_{\text{this is } \boldsymbol{n}} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

or

$$\boldsymbol{n} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0.$$

This is why Eq. (24) is called the "normal form", because it involves the normal vector $\boldsymbol{n}$.

Example 80. Find the normal equation of the plane through the point $\boldsymbol{p} = (2, 4, -1)$ with normal vector $\boldsymbol{n} = \langle 2, 3, 4\rangle$. Find the intercepts and sketch the plane.

Solution: By Eq. (24), with $A = 2$, $B = 3$, $C = 4$, we have:

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0.$$

We are done. We could also write this as:

$$2x + 3y + 4z = 12. \tag{25}$$

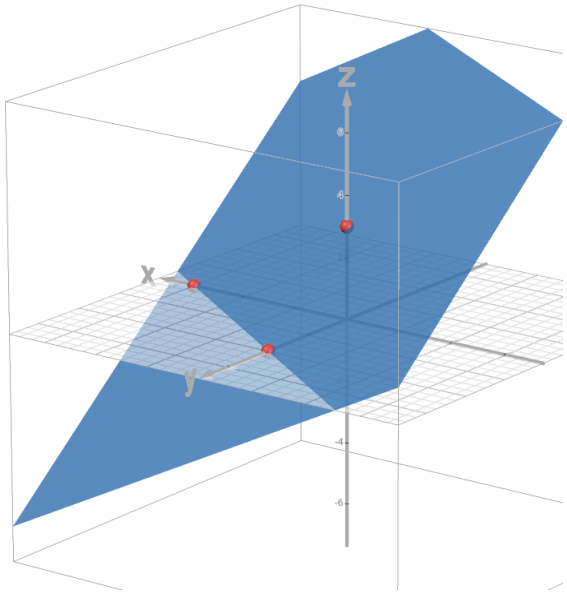
To find the intercepts, we consider what happens when we set two of the variables x, y, z to be equal to zero at a time:

- To find the x-interecept, set $y = z = 0$ in Eq. (25). In that case, we get $x = 6$. Hence the x-intercept is $(6, 0, 0)$.

- To find the y-interecept, set $x = z = 0$ in Eq. (25). In that case, we get $y = 4$. Hence the y-intercept is $(0, 4, 0)$.
- To find the z-interecept, set $x = y = 0$ in Eq. (25). In that case, we get $z = 3$. Hence the z-intercept is $(0, 0, 3)$.

Recall that any two points uniquely determine a line. A similar fact holds for planes:
Fact: A plane is uniquely determined by any three points (as long as they don't all lie in a line).

So the plane can be plotted as follows:



(see: `https://www.desmos.com/3d/brbgxh7qz7`)

End of Example 80. □

20 2026-10-09 | Week 07 | Lecture 20

Topic of this lecture: *Planes*
Reading assignment: *12.5*

20.1 Plane geometry examples

Note: the solutions to the example problems here are somewhat simpler than the solutions I presented in class.

Example 81. Consider the three points $P = (1, 3, 2)$, $Q = (3, -1, 6)$, and $R = (5, 2, 0)$. These three points are not colinear (i.e., they don't all lie in a line), and hence there is a single, unique plane passing through all three of them. Find the equation of the plane passing through P, Q, and R.

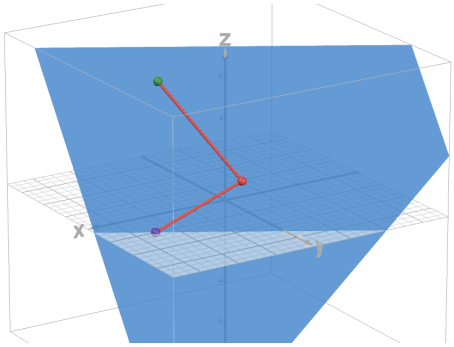
Solution:

- $\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$
- $\overrightarrow{PR} = \langle 4, -1, -2 \rangle$

We need a normal vector $\boldsymbol{n}$ which is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. We can use the cross product. Let

$$\begin{aligned}
\boldsymbol{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\
&= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \\
&= \vec{\imath}(8+4) - \vec{\jmath}(-4-16) + \vec{k}\,(-2+16) \\
&= 12\vec{\imath} + 20\vec{\jmath} + 14\vec{k} \\
&= \langle 12, 20, 14 \rangle
\end{aligned}$$

Here's a picture:



(see `https://www.desmos.com/3d/sqdoxhvsng`)

End of Example 81. □

Example 82. The two planes

$$3x - 6y - 2z = 15, \qquad 2x + y - 2z = 5$$

intersect to form a line. Find a vector parallel to this line.

Solution: By inspection of the two above equations, the normal vectors for the two planes are

$$\boldsymbol{n}_1 = \langle 3, -6, -2 \rangle, \qquad \boldsymbol{n}_2 = \langle 2, 1, -2 \rangle.$$

If $\boldsymbol{w}$ is a vector which is parallel to the line of intersection then it must be orthogonal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ [think about why].

Therefore we may take $\boldsymbol{w}$ to be the cross product of the two normal vectors:

$$\boldsymbol{w} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = ... = \begin{bmatrix} 14 \\ 2 \\ 15 \end{bmatrix}.$$

End of Example 82. □

Example 83. Consider the two planes from the previous example:

$$3x - 6y - 2z = 15, \qquad 2x + y - 2z = 5. \tag{26}$$

These intersect to form a line. Find the parameteric equations for this line.

Solution: We computed the direction vector $\langle 14, 2, 15 \rangle$ in the previous example, so we need only a point on the line.

Set $z = 0$ in both equations of Eq. (26). This gives

$$\begin{aligned} 3x - 6y &= 15 \\ 2x + y &= 5 \end{aligned}$$

Solving for x and y gives

$$x = 3, \qquad y = -1$$

Therefore the point

$$(x, y, z) = (3, -1, 0)$$

lies on the line.

Hence we may write the line in *vector form* as

$$\boldsymbol{r}(t) = \begin{bmatrix} 14 \\ 2 \\ 15 \end{bmatrix} t + \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \qquad -\infty < t < \infty.$$

Writing this in *parametric form* gives

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

for $-\infty < t < \infty$.

End of Example 83. □

21 2026-10-12 | Week 08 | Lecture 21

Topic of this lecture: *planes*
Reading assignment: *chapter 12.5*

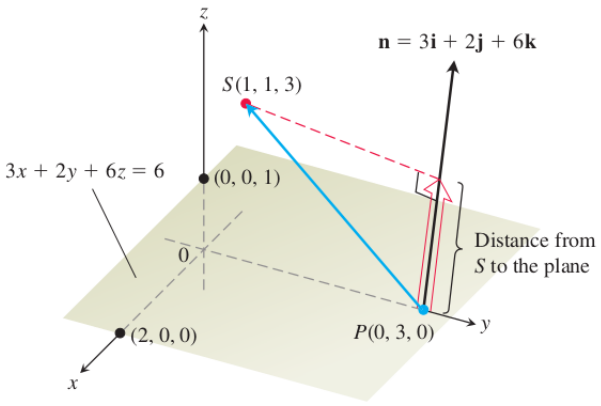
21.1 Plane geometry examples (continued)

Example 84. Find the distance from the point $S = (1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution:

Step 1. Find *any* point on the plane. For example, $P = (0, 3, 0)$ lies on the plane.

Step 2. Draw it



Step 3. Observe that the distance is length of the projection of $\overrightarrow{PS}$ along $\vec{n}$:

$$\text{Projection} = \left(\frac{\overrightarrow{PS} \cdot \vec{n}}{|\vec{n}|}\right)\left(\frac{\vec{n}}{|\vec{n}|}\right)$$

This has length

$$\text{length} = \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|}$$

which is

$$\begin{aligned}\text{length} &= \frac{\langle 1, -2, 3\rangle \cdot \langle 3, 2, 6\rangle}{\sqrt{9 + 4 + 36}} \\ &= \frac{17}{\sqrt{49}} \\ &= \frac{17}{7}.\end{aligned}$$

End of Example 84. □

Example 85. Find the equation of the plane P which passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $5x + 4y - 2z = 0$.

Solution: We need to find the normal vector $\boldsymbol{n}$ to P. A vector joining the two points is

$$\boldsymbol{v} = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}.$$

We know that $\boldsymbol{n}$ must be orthogonal to both $\boldsymbol{v}$ and to

$$\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}.$$

Hence we may take

$$\boldsymbol{n} = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix} \times \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} = \ldots = \begin{bmatrix} -6 \\ 22 \\ 29 \end{bmatrix}.$$

Therefore an equation of P is

$$-6(x-0) + 22(y+2) + 29(z-5) = 0.$$

End of Example 85. □

Example 86. Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution: $(2/3, 2, 0)$

End of Example 86. □

Example 87. Find the angle between the two planes $x + y + z = 1$ and $x - 2y + 3x = 1$.

Solution: It will be enough to find the angle between the two normal vectors. We can use the dot product. We'll get $\cos(\theta) = \frac{2}{\sqrt{41}}$. The angle with this cosine is about 72 degrees.

End of Example 87. □

Example 88. Consider the plane H which passes through the three points

$$P = (1, 4, 6), \qquad Q = (-2, 5, -1), \qquad R = (1, -1, 1).$$

Find a vector perpendicular to H.

End of Example 88. □

Example 89. Find the area of the triangle with vertices

$$P = (1, 4, 6), \qquad Q = (-2, 5, -1), \qquad R = (1, -1, 1).$$

End of Example 89. □

22 2026-10-14 | Week 08 | Lecture 22

We did practice exam in class today.

23 2026-10-19 | Week 09 | Lecture 23

Topic of this lecture: *cylinders and quadric surfaces*
Reading assignment: *chapter 12.6*

23.1 Cylinders and quadric surfaces

Definition 90 (Level set, zero set)**.** Let $f = f(x, y, z)$ be a function of three variables. A ***level set*** of f is an set of the from

$$\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\} \tag{27}$$

where $c \in \mathbb{R}$. A level set with $c = 0$ is called the ***zero set*** of f.

Level sets for functions of other numbers of variables are defined similarly.

Notation 91. For brevity, we usually omit the set notation, and instead of Eq. (27), we just write

$$f(x, y, z) = 0.$$

Many shapes and surfaces are defined by level sets.

Example 92. In 2 dimensions,

- $y - x^2 = 0$ is a parabola
- $2x + y - 3 = 0$ is a line
- $x^2 + y^2 = 0$ is a circle

In 3 dimensions,

- $x^2 + y^2 - 1 = 0$ is the unit cylinder
- $x^2 + y^2 + z^2 - 9 = 0$ is a sphere with radius 3.
- $z = x^2$ is a parabolic cylinder
- $Ax + By + Cz - D = 0$ is a plane

End of Example 92. □

Definition 93 (Quadric surface)**.** A general degree-2 polynomial in the variables x, y, z takes the form

$$f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J,$$

where $A, B, C, \ldots, J \in \mathbb{R}$. The zero set of such a polynomial is called a ***quadric surface***.

There are many different quadric surfaces, so we can't do them all, but we'll do some special cases that will provide a rich source of examples.

Before we start we recall some things from 2-dimensional space (see chapter 11.6 in the texbook):

Proposition 94 (Conic sections in 2D)**.**

1. ***Ellipse****:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This ellipse is centered at $(0, 0)$ *and passes through the points* $(\pm a, \pm b)$.

2. *Left-right* ***hyperbola****:*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The vertices are $(\pm a, 0)$ *and asymptotes* $y = \pm \frac{b}{a} x$.

3. *Up-down* ***hyperbola****:*

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The vertices $(0, \pm b)$ *and asymptotes* $y = \pm \frac{a}{b} x$*.*

Definition 95. The intersection of a set with a plane is called a ***cross section***, or a ***slice***.

Example 96 (Quadric Form, Special case #1: Elliptic Paraboloid)**.** Consider level set

$$4x^2 + y^2 - z = 0, \tag{28}$$

or equivalently

$$z = 4x^2 + y^2.$$

Question: What is this shape? Consider its cross-sections:

- Consider the plane $x = 0$. Intersecting the plane with Eq. (28) gives

  $$z = y^2.$$

  So this cross section is a parabola.

- Now consider the plane $x = k$, where $k \in \mathbb{R}$ is a constant. This plane is parallel to to the yz-plane. The intersection is

  $$z = y^2 + 4k^2$$

  Thus every cross-section of our surface with a plane parallel to the yz-plane is a parabola.

- The cross-sections with $y = k$ also give parabolas (you should check this).

- The plane $z = k$ is parallel to the xy-axis. Cross sections with this plane are different:

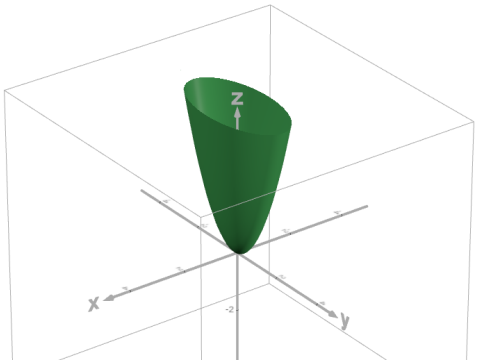
  $$4x^2 + y^2 = k$$

  When $k < 0$, there are no real numbers which satisfy the equation. But when $k > 0$, this is an ellipse:

  $$\frac{x^2}{(k/4)} + \frac{y^2}{k} = 1.$$

  As k increases, this elipse gets bigger.

Thus, the shape is



End of Example 96. □

Example 96 is an example of an ***elliptic paraboloid***, which (up to translation and rotation) has equation

$$z = Ax^2 + By^2.$$

24 2026-10-21 | Week 09 | Lecture 24

Topic of this lecture: *quadric surfaces*
Reading assignment: *chapter 12.6. Please start 13.1*

24.1 Quadric surfaces

Example 97 (Quadric Form, Special case #2: Hyperbolic Paraboloid)**.** Consider the surface defined by

$$z = \frac{x^2}{9} - \frac{y^2}{4}. \tag{29}$$

Question: What shape is this?
Again, consider the cross-sections.

- Cross section $x = k$:

$$z = \frac{k^2}{9} - \frac{y^2}{4}.$$

  This is a downward-opening parabola.

- Cross section $y = k$:

$$z = \frac{x^2}{9} - \frac{k^2}{4}$$

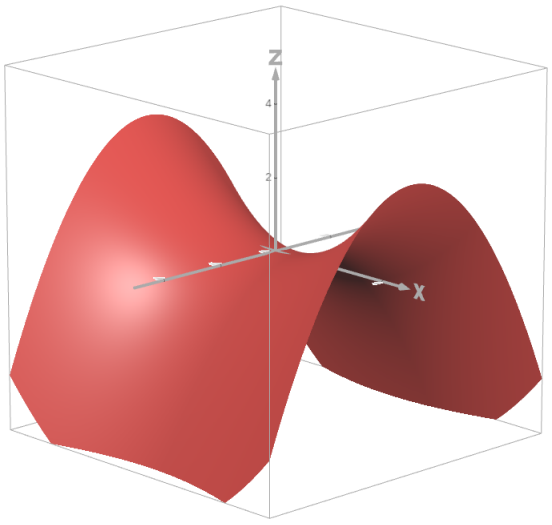
  This is an upward-opening parabola

- Cross section $z = k$:

$$k = \frac{x^2}{9} - \frac{y^2}{4}$$

  This is hyperbola. How it opens depends on whether $k > 0$ or $k < 0$

Example 97 looks like a pringle.



Check it out on Desmos (along with some cross sections) here:

`https://www.desmos.com/3d/xuaxssutef`

End of Example 97. □

Example 97 is an example of a ***hyperbolic paraboloid***, which (up to translation and rotation), has equation

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Important fact: the origin is a saddle point.

Example 98 (Quadric Form, Special case #3: Ellipsoid)**.** Consider the quadric form

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0 \tag{30}$$

What is this shape?

To investigate, we consider cross sections.

- Fix a number $k \in \mathbb{R}$, and consider the plane $z = k$. This plane is parallel to the xy-plane. Intersect the plane $z = k$ and ther surface Eq. (30). That gives

  $$x^2 + \frac{y^2}{9} + \frac{k^2}{4} - 1 = 0$$

  or

  $$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}.$$

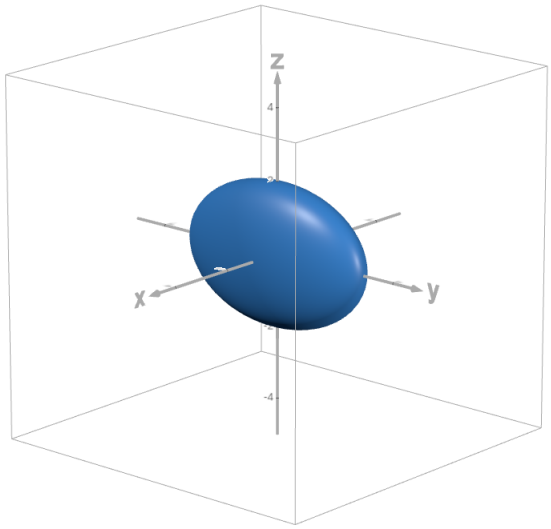
  or

  $$\frac{x^2}{1 - \frac{k^2}{4}} + \frac{y^2}{9(1 - \frac{k^2}{4})} = 1.$$

  As long as $k^2 < 4$, this is the equation of an ellipse by Proposition 94.

- The cross sections with the plane $x = k$ also gives an ellipse.
- The cross sections with the plane $y = k$ also gives an ellipse.

All of the traces are ellipses. (You should check the last two). This suggests an egg-like shape:



End of Example 98. □

Example 98 is an example of an ***ellipsoid***, which has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

25 2026-10-23 | Week 09 | Lecture 25

Topic of this lecture: *vector-valued function*
Reading assignment: *chapter 13.1*

25.1 Vector functions and space curves

We are interested in describing motion along a curve durin a time interval. A bug flying around during a time interval I has coordinates

$$\boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \qquad t \in I.$$

The functions $x(t), y(t), z(t)$ are assumed to be real-valued functions[1], and are called ***component functions***. The function $\boldsymbol{r}$ is called a ***vector-valued function***, aka a ***vector function***. The points

$$\{(x(t), y(t), z(t)) : t \in I\}$$

make up a ***curve*** in space, i.e., the ***path*** of the bug.

We have already seen examples of vector-valued function.

Example 99 (A vector-valued function)**.** A dragonfly takes off from a leaf at position $(2, 4, -3)$ traveling in a straight line with velocity $\langle 1, -5, 4\rangle$. Then the dragonfly's position at time $t \geq 0$ is

$$\boldsymbol{r}(t) = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} t + \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} t+2 \\ -5t+4 \\ 4t-3 \end{bmatrix}, \qquad t \geq 0.$$

End of Example 99. □

Limits, derivatives, integrals of vector-valued functions are defined *componentwise*:

- Limits:
$$\lim_{t\to a} \boldsymbol{r}(t) = \begin{bmatrix} \lim_{t\to a} f(t) \\ \lim_{t\to a} g(t) \\ \lim_{t\to a} h(t) \end{bmatrix}$$
- Derivatives:
$$\boldsymbol{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$
- Integrals:
$$\int_a^b \vec{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

Example 100. Consider

$$\mathbf{r}(t) = \begin{bmatrix} 2+t^3 \\ \frac{\sin(t)}{t} \\ te^{-t} \end{bmatrix}, \qquad t \in \mathbb{R}.$$

Question: What is $\lim_{t\to 0} \boldsymbol{r}(t)$, and what is $\boldsymbol{r}(t)$?

$$\lim_{t\to 0} \mathbf{r}(t) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{r}'(t) = \begin{bmatrix} 3t^2 \\ \frac{\cos(t)t - \sin(t)}{t^2} \\ e^{-t} - te^{-t}. \end{bmatrix}$$

[1] Recall that a ***real-valued function*** is one whose range is a subset of $\mathbb{R}$.

End of Example 100. □

We say that a vector function $\boldsymbol{r}$ is ***continuous*** on I if each of its component functions are continuous on I. Similarly, $\boldsymbol{r}(t)$ is ***differentiable*** if each of its component functions are. We say the curve traced by $\boldsymbol{r}$ on the interval I is ***smooth*** if $\boldsymbol{r}'(t)$ is continuous on I and

$$\boldsymbol{r}(t) \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{for all } t \in I$$

As in Definition 23, this condition excludes corners and cusps.

Remark 101 (Interpretation)**.** *We regard $\boldsymbol{r}(t)$ as the **position** of the bug at time t. Its intantaneous **velocity** is $\boldsymbol{r}'(t)$. It's **acceleration** is $\boldsymbol{r}''(t)$. The normalized velocity is*

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|},$$

*which is a unit vector identifying the direction that the bug is moving at time t. We call $\boldsymbol{T}(t)$ the **unit tangent vector**.*

End of Remark 101. □

Problem 102. Find the velocity, speed and acceleration of a particle whose motion in space is given by

$$\boldsymbol{r}(t) = \left\langle 2\cos(t), 2\sin(t), 5\cos^2(t) \right\rangle$$

Solution: The velocity is

$$\begin{aligned} \boldsymbol{v}(t) &= \boldsymbol{r}'(t) \\ &= \langle -2\sin(t), 2\cos(t), -10\cos(t)\sin(t) \rangle \\ &= \langle -2\sin(t), 2\cos(t), -5\sin(2t) \rangle \end{aligned}$$

$$\boldsymbol{a}(t) = \boldsymbol{r}''(t) = \langle -2\cos(t), -2\sin(t), -10\cos(2t) \rangle$$

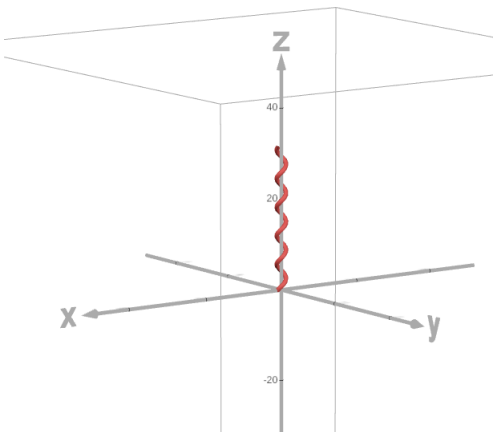
The speed is

$$|\boldsymbol{r}'(t)| = \sqrt{4 + 25\sin^2(2t)}$$

Example 103 (Helix)**.** A moth flies spirals upwards towards a light. Its trajectory is

$$\boldsymbol{r}(t) = \langle \cos(t), \sin(t), t \rangle, \qquad 0 \leq t \leq 10\pi.$$

Suppose that the coordinate axes are measured in feet and time is measured in seconds.



The velocity of the moth at time t is

$$\boldsymbol{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle,$$

and its speed is

$$|\boldsymbol{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

The unit tangent vector is

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \left\langle -\frac{\sin(t)}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

This tells us the direction the moth is facing at time t.

Question: How far did the moth fly? Using the formula

$$\boxed{(\text{Arc length}) = \int_a^b |\boldsymbol{r}'(t)| dt.}$$

we have

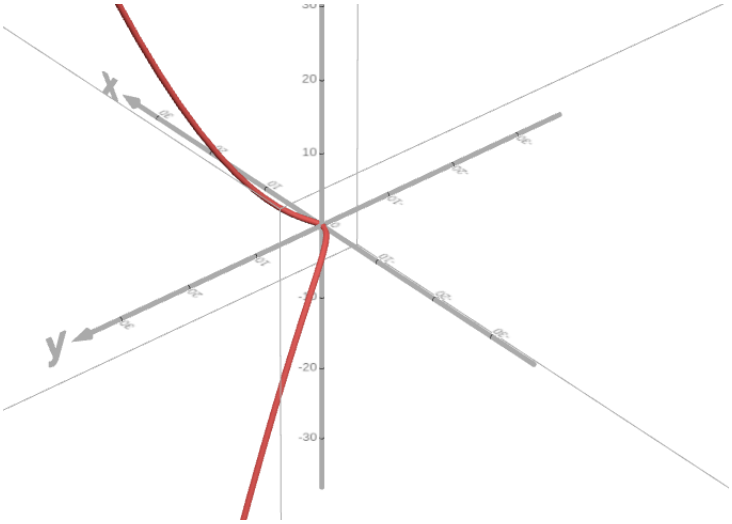
$$(\text{Arc length}) = \int_0^{10\pi} \sqrt{2} dt = 10\pi \cdot \sqrt{2} \approx 44 \text{ meters}.$$

The moth traveled $10\pi \cdot \sqrt{2} \approx 44$ feet.

This makes sense: after all, the moth traveled at a constant velocity of $\sqrt{2}$ meters per second for 10π seconds. So of course it traveled $10\pi\sqrt{2}$ meters.

End of Example 103. □

Example 104 (Twisted Cubic)**.** Let $\boldsymbol{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle$, for $t \in \mathbb{R}$.



The velocity is

$$\boldsymbol{r}'(t) = \left\langle 2, 2t, t^2 \right\rangle.$$

The speed is

$$|\boldsymbol{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2.$$

So the particle is accelerating as $t \to \infty$ or $t \to -\infty$.

The distance traveled by the particle from $t = 0$ to $t = 3$ is

$$\begin{aligned}
\text{Arc length} &= \int_0^3 |\boldsymbol{r}'(t)| dt \\
&= \int_0^3 (2 + t^2) dt \\
&= \left[2t + \frac{t^3}{3} \right]_0^3 \\
&= (6 + 9) - (0 + 0) \\
&= 15.
\end{aligned}$$

The direction of the particle, without respect to the speed is given by the unit tangent vector

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \left\langle \frac{2}{2 + t^2}, \frac{2t}{2 + t^2}, \frac{t^2}{2 + t^2} \right\rangle$$

End of Example 104. □

26 2026-10-26 | Week 10 | Lecture 26

Question: *How do we quantify how curvy or straight a curve is?*
Reading assignment: *Chapter 13.4*

26.1 Curvature

Recall the ***unit tangent vector*** of a parametrized curve $\mathbf{r}(t)$ is the vector

$$\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

In other words, $\mathbf{T}$ is the normalized velocity.

The **curvature** of a curve at a point (defined precisely below) is a nonnegative number which measures how quickly the the curve changes direction at that point. Essentially, it works like this

- straight line = zero curvature
- wide turns = low curvature
- sharp turns = high curvature

The direction is given by $\mathbf{T}(t)$. When a curve is fairly straight, $\mathbf{T}(t)$ shouldn't change much over time. But it should change direction quickly when the curve bends or twists sharply.

There is a complication: the rate that $\mathbf{T}(t)$ changes depends on the speed of the particle (or bug) that traces out the curve. If you speed up the particle that traces out the curve, then $\mathbf{T}(t)$ may change faster. Therefore, what we want is *not* the rate of change of $\mathbf{T}$ with respect to *time*, but rather with respect to *distance traveled*.

If the bug starts moving at time a and stops at time t, then the distance it travels is given by the following function, called the ***arc length function***:

$$s(t) := \int_a^t |\mathbf{r}'(u)| du, \quad t \geq a.$$

Thus we define curvature in the following way:

Definition 105 (Curvature)**.** The ***curvature***, denote κ, is defined as

$$\kappa := \left| \frac{d\mathbf{T}}{ds} \right|,$$

where s is the arc length function.

Question: The above definition is not very useful for doing computations. How do we compute κ?

We can derive a more useful formula in the following way. By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Changing notation,

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{ds} s'(t).$$

Rearranging gives

$$\frac{d\mathbf{T}}{ds} = \frac{1}{s'(t)} \mathbf{T}'(t)$$

By the fundamental theorem of calculus, $s'(t) = |\mathbf{r}'(t)|$. Hence

$$\frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|}$$

Taking absolute valus of both sides gives the following important **curvature formula:**

$$\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}} \tag{31}$$

Example 106 (Curvature of a line)**.** Consider a line in $\mathbb{R}^3$ with vector form

$$\mathbf{r}(t) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} t + \begin{bmatrix} x_0 \\ y_0 \\ z_0, \end{bmatrix}, \qquad -\infty < t < \infty.$$

where $v_1, v_2, v_3, x_0, y_0, z_0 \in \mathbb{R}$ are fixed real numbers.
Question: What is the curvature of the line?

We want to apply Eq. (31). Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Since $\mathbf{r}'(t) = \mathbf{v}$, we have

$$\mathbf{T}(t) = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{v_1}{|\mathbf{v}|}, \frac{v_2}{|\mathbf{v}|}, \frac{v_3}{|\mathbf{v}|} \right\rangle.$$

Note that the right-hand side does not depend on t. Therefore

$$\mathbf{T}'(t) = \langle 0, 0, 0 \rangle, \qquad \text{so} \qquad |\mathbf{T}'(t)| = 0 \text{ for all } t.$$

Therefore by Eq. (31),

$$\kappa(t) = 0, \qquad \text{for all } t \in [0, 2\pi].$$

This makes sense: a straight line never changes direction. So of course under any reasonable definition of "curvature", the curvature of a straight line should always be zero.

End of Example 106. □

Example 107 (Curvature of a circle)**.** Consider a circle of radius $a > 0$. We can parametrize this with

$$\mathbf{r}(t) = \begin{bmatrix} a\cos(t) \\ a\sin(t) \end{bmatrix}, \qquad 0 \le t \le 2\pi.$$

What "should" the curvature be? It should be

- constant
- inversely related to a

Indeed, using Eq. (31) to compute the curvature gives

$$\kappa(t) = \frac{1}{a} \quad \text{for all } t \in [0, 2\pi].$$

[Do this calculation. For an alternative expanation, see page 784 of the textbook.]

End of Example 107. □

27 2026-10-28 | Week 10 | Lecture 27

Topic of this lecture: *curvature, projectile motion*
Reading assignment: *13.2, 13.5*

27.1 A useful curvature formula

Proposition 108 (A useful formula for curvature)**.**

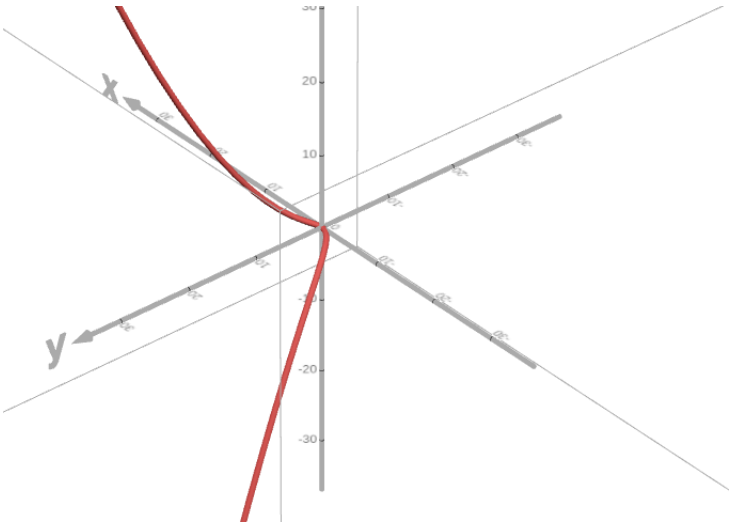
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

See p.791 of the text for an explanation.

Example 109 (twisted cubic curvature)**.** Let

$$\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle, \qquad -\infty < t < \infty.$$

This is an example of a twisted cubic, which we saw in Example 104. Recall this has plot



From this picture, we might reasonably conclude that the curvature should be high near the origin (when t is near zero) and then it should be low far away from the origin (when t is large) because there the graph appears to straighten out. Let's try to compute the curvature:

We already showed that

$$\mathbf{r}'(t) = \langle 2, 2t, t^2 \rangle$$

and so

$$|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2$$

Therefore the unit tangent vector (i.e., the normalized velocity of the particle) is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \frac{2}{2+t^2}, \frac{2t}{2+t^2}, \frac{t^2}{2+t^2} \right\rangle.$$

Using Eq. (31),

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

But this is going to be a lot of work to take all those derivatives. So instead, let's use Proposition 108:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

- *Numerator:*

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} = \left\langle 2t^2, -4t, 4 \right\rangle.$$

  And this implies that

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4t^4 + 16t^2 + 16} = 2\sqrt{t^4 + 4t^2 + 4} = 2\sqrt{(t^2+2)^2} = 2(t^2+2)$$

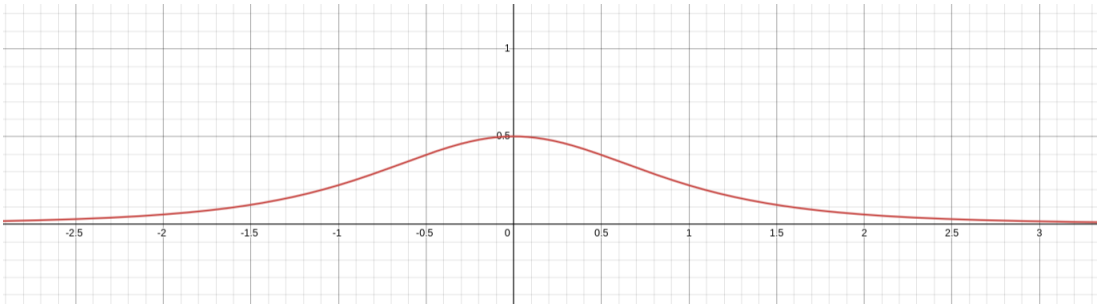
- *Denominator:* Since $\mathbf{r}'(t) = \left\langle 2, 2t, t^2 \right\rangle$, we have

$$|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = t^2 + 2$$

Therefore

$$\kappa(t) = \frac{2}{(2+t^2)^2}$$

This is pretty complicated-looking function, but when we graph $\kappa(t)$, we see that it's actually really simple:



As we expected, the curvature has a bump near zero, and then asymptotically approaches zero as $|t| \to \infty$.

End of Example 109. □

27.2 Motion in space: velocity, acceleration, projectile motion

Example 110. A stone is dropped form a 1600 foot high building.

(a) Find, as functions of time, its position and velocity.

(b) When does it hit the ground?

(c) How fast is it going when at that time?

You may assume that acceleration due to gravity is 32ft/sec^2.
Part (a): Let

$$s(t) = \text{position}$$
$$v(t) = \text{velocity}$$
$$a(t) = \text{acceleration}$$

Then $a(t) = \frac{dv}{dt} = -32$ (negative because the direction is downward). So

$$v(t) = \int a(t)dt = \int -32dt = -32t + C_0$$

Since the stone was dropped, not thrown, its initial velocity was zero, i.e.,

$$v(0) = 0.$$

Therefore $C_0 = 0$, and it follows that

$$\boxed{v(t) = -32t}$$

Next we integrate velocity to find position:

$$s(t) = \int v(t)dt = \int -32tdt = -16t^2 + C_1$$

Since the stone started at height 1600, we have

$$s(0) = 1600.$$

Therefore

$$100 = -16(0)^2 + C_1$$

so $C_1 = 1600$. Thus,

$$\boxed{s(t) = -16t^2 + 1600.}$$

Part (b): Let t^* be the time that the stone hits the ground. It must be the case that

$$s(t^*) = 0.$$

By our formula for $s(t)$, we have

$$0 = -16(t^*)^2 + 1600$$

and solving for t^*, we conclude that $\boxed{\text{the stone hits the ground at time } t^* = 10}$.

Part (c): This part is asking what the speed is at time $t^* = 10$. At that time, the velocity is $v(10) = -32(10) = -320$ feet per second.

Therefore $\boxed{\text{the stone is traveling at 320 feet per second when it hits the ground.}}$

End of Example 110. □

We want to do the same thing in 3 dimensions. We will use the vector equations

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(u)du \quad \text{and} \quad \mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} a(u)du$$

or equivalently,

$$\mathbf{r}\,'(t) = \mathbf{v}(t) \quad \text{and} \quad \mathbf{v}\,'(t) = \mathbf{a}(t)$$

Example 111. Find the velocity, acceleration, and speed of a particle with position vector $r(t) = \langle t^2, e^t, te^t \rangle$

End of Example 111. □

Example 112. A moving particle starts at initial position $r(0) = \langle 1, 0, 0 \rangle$. It has initial velocity $v(0) = \langle 1, -1, 1 \rangle$. Its acceleration is $a(t) = \langle 4t, 6t, 1 \rangle$. Find its velocity and position at time t.

End of Example 112. □

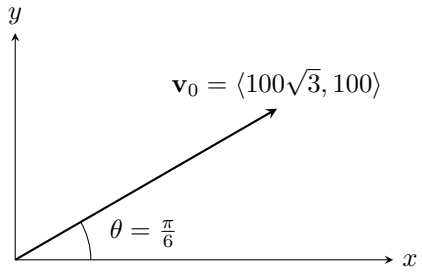
28 2026-10-30 | Week 10 | Lecture 28

Topic of this lecture: *Projectile Motion*
Reading assignment: *13.2, 14.1, 14.2*

Example 113. A cannon is fired with initial velocity

$$\mathbf{v}_0 = \left\langle 100\sqrt{3}, 100 \right\rangle.$$

This corresponds to being fired at an intial angle of elevation $\theta = \frac{\pi}{6}$ with the cannonball traveling at an initial speed of 200 meters per second.



Assume the canonball is fired from an elevation of 3 meters, and that downward acceleration due to gravity is

$$\mathbf{a}(t) = \langle 0, -g \rangle,$$

where $g = 9.8\mathtt{m}/\mathtt{s}^2$.

Some questions

- How long is the cannonball airborne?
- How far does the canonball travel?
- How fast is it going when it hits the ground?
- Can we characterize the trajectory of the canonball?

Solution: Let $\mathbf{v}(t)$ be the velocity of the cannonball at time t. Then

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(u)du \\
&= \left\langle 100\sqrt{3}, 100 \right\rangle + \langle 0, -gt \rangle \\
&= \left\langle 100\sqrt{3}, 100 - gt \right\rangle.
\end{aligned}$$

Let $\mathbf{r}(t)$ be the cannonball's position at time t. Then

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(u)du \\
&= \langle 0, 3 \rangle + \left\langle \int_0^t 100\sqrt{3}dt, \int_0^t (100 - gt)dt \right\rangle \\
&= \langle 0, 3 \rangle + \left\langle 100\sqrt{3}t, 100t - \frac{1}{2}gt^2 \right\rangle \\
&= \left\langle 100\sqrt{3}t, 3 + 100t - \frac{1}{2}gt^2 \right\rangle
\end{aligned}$$

Plugging in $g = 9.8$, the position at t seconds is

$$\begin{cases} x(t) = 100\sqrt{3}t \\ y(t) = 3 + 100t - 4.9t^2 \end{cases} \tag{32}$$

The cannonball is airborne as long as $y(t) > 0$. To find the time when $y(t) = 0$, we need to solve the quadratic equation

$$-4.9t^2 + 100t + 3 = 0.$$

This has two solutions:

$$t \approx -0.4, \qquad t \approx 20.8.$$

Since $t = -0.4$ doesn't make sense for this problem, we ignore it. We conclude the canonball remains airborne for about 20.8 seconds. At $t = 20.8$, it hits the ground; when that happens, its x-value is

$$x(20.8) \approx 3607 \text{ meters},$$

which is how far horizontally the cannonball went.

At the moment of impact, its velocity is

$$\mathbf{v}(20.8) = \begin{bmatrix} 100\sqrt{3} \\ 100 - 9.8(20.8) \end{bmatrix} = \begin{bmatrix} 100\sqrt{3} \\ -103.84 \end{bmatrix}$$

so its speed is

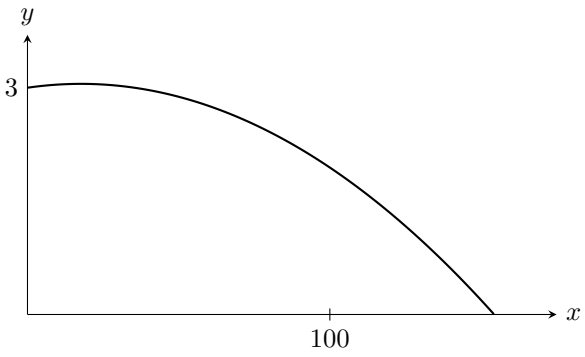
$$|\mathbf{v}(20.8)| = \sqrt{(100\sqrt{3})^2 + (-103.84)^2} \approx 202 m/s$$

So it's slowed down a lot.

We can eliminate t in Eq. (32) to obtain the equation

$$y = -\frac{4.9}{30000}x^2 + \left(\frac{1}{100\sqrt{3}}\right)x + 3$$

which is a parabola. Thus, the cannonball traveled in a parabolic arc:



The axes are not to scale.

End of Example 113. □

28.1 Functions of several variables

Chapters 14.1, 14.2

Definition 114. Suppose D is a set of n-tuples of real numbers $(x_1, \ldots, x_n)$. A ***real-valued function*** f on D is a rule that assigns a real number

$$w = f(x_1, \ldots, x_n)$$

to each element in D. The set D is the function's ***domain***. The set of w-values taken on by f is the function's ***range***. The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***indepenedent variables*** $x_1, \ldots, x_n$. We also call these the ***input variables*** and w the ***output variable***.

Explain interior and boundary points for functions of two variables.

Remark 115. *If $n = 2$, we usually call the input variables x and y, and the output variable z. If $n = 3$, we usually call the input variables x, y, z and the output variable w.*

End of Remark 115. □

Example 116. The function

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

has domain

$$D = \left\{(x, y) \in \mathbb{R}^2 : x \neq 1\right\}$$

End of Example 116. □

Example 117. Let

$$f(x, y) = x \log\left(x^2 - y\right)$$

What is the domain of f?

We can find the domain D of f by recalling that $\log(u)$ is defined only for $u > 0$. Thus, we require that $x^2 - y > 0$, or $y < x^2$. [Plot this].

End of Example 117. □

29 2026-11-02 | Week 11 | Lecture 29

Topic of this lecture: *What does it mean for a real-valued function of two variables $f(x,y)$ to be continuous?*
Reading assignment: *chapter 14.2*

29.1 Limits and Continuity

We began with a discussion of what it means for a function of a single variable to be continuous or discontinuous, and the ways that such a function can be discontinuous (point discontinuity, jump discontinuity, or oscillation).

To be precise, we have the following definition

Definition 118. Let $f(x)$ be a real-valued function of a single variable whose domain is $\mathbb{R}$. We say that f ***is continous at*** a if and only if

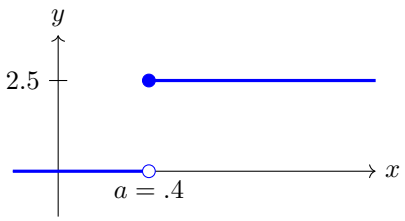
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

In words, the left- and right-hand limits must agree, and they must equal the function value. (Note that it's not obvious how to generalize this definition to funcions of multiple variables, but that's our task.)

One issue that can arise is that the left and right-hand limits might not agree, such as in the following example:

$$g(x) = \begin{cases} 0 & : \quad x < .4 \\ 2.5 & : \quad x \geq .4 \end{cases}$$

which has graph



in this case, in this case, we have

$$\lim_{x\to 0.4^-} g(x) = 0 \neq 2.5 = \lim_{x\to 0.4^+},$$

and hence

$$\lim_{x\to 0.4} g(x) \text{ does not exist.}$$

Definition 119. Let $f(x,y)$ be a real-valued function of two variables with domain, and fix $(a,b) \in \mathbb{R}^2$. Consider a sequence of points

$$(x_1,y_1),(x_2,y_2),(x_3,y_3),\ldots$$

in the domain of f such that the points are getting closer to (a,b) in the sense that

$$\lim_{n\to\infty} \text{dist}((x_n,y_n),(a,b)) = 0.$$

If for every such sequence we have

$$\lim_{n\to\infty} f(x_n,y_n) = L,$$

then we say that L ***is the limit of*** $f(x,y)$ ***as*** $(x,y)\to(a,b)$, and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

On the other hand, if there are two different sequences

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$$

and

$$(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2), (\tilde{x}_3, \tilde{y}_3), \ldots$$

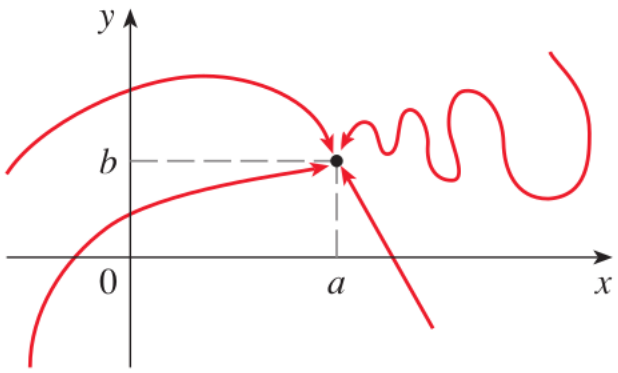
with

$$\lim_{n\to\infty} f(x_n, y_n) \neq \lim_{n\to\infty} f(\tilde{x}_n, \tilde{y}_n)$$

then we say that the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ ***does not exist (DNE)***.

Remark 120. *The above definition is analogous to the limit of a single-variable function not existing if the left and right hand limits don't agree.*

The complication is that, now, instead of having only two paths of approach (left and right) we now have infinitely many paths by which (x, y) can approach (a, b):



End of Remark 120. □

It is hard to show that a function of two variables is continous at a point, because doing so requires considering all possible paths of approach. So there are lots of ways things can go wrong. A classic example:

Example 121. Let

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Does the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist?

Answer: No, for the following reason:

- Approach along the x-axis (i.e., when $y = 0$) and you get

$$f(x, y) \to 1 \text{ as } (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

- Approach along the y-axis, and you get -1.

Since these are not equal, the limit does not exist.

End of Example 121. □

Example 122. Does the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

exist?

No. Consider the path along the line $y = mx$:

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+(mx)^2} = \frac{m}{1+m^2}.$$

This above limit depends on the the slope m that you pick: if $m = 0$ the limit is $10/101$. Therefore the overall limit does not exist.

End of Example 122. □

Example 123. Let

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Does the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exist?

No. First, let's try approaching along any line $y = mx$. In that case we get

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0) \\ y=mx}} f(x,y) &= \lim_{x\to 0} f(x,mx) \\
&= \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} \\
&= \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} \\
&= 0
\end{aligned}$$

Then, we try doing along the parabola $x = y^2$:

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0) \\ x=y^2}} f(x,y) &= \lim_{y\to 0} f(y^2,y) \\
&= \lim_{y\to 0} \frac{y^4}{y^4+y^4} \\
&= \frac{1}{2}.
\end{aligned}$$

The limits along the different paths are not equal. Therefore the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

End of Example 123. □

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Topic of this lecture: *What does it mean for a real-valued function of two variables $f(x,y)$ to be continuous?*
Reading assignment: *chapter 14.2*

Definition 124. A real-valued function of two variables f is ***continuous at*** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ exists and equals } f(a,b).$$

A function is ***continuous*** if it is continuous at every point in its domain.

Example 125. Let

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 0 & : \quad (x,y) = (0,0) \end{cases}$$

Question. Is $f(x,y)$ is continuous at $(0,0)$?

Answer. Yes. We will show that that limit is zero, which equals $f(0,0)$. We will use the squeeze theorem. First observe that

$$\begin{aligned} |f(x,y)| &= \left| \frac{3x^2y}{x^2+y^2} \right| \\ &\leq 3|y| \left(\frac{x^2}{x^2+y^2} \right) \\ &\leq 3|y| \\ &\leq 3\sqrt{x^2+y^2} \\ &= 3\text{dist}((x,y),(0,0)) \end{aligned}$$

Therefore

$$0 \leq |f(x,y)| \leq 3\text{dist}((x,y),(0,0))$$

As $(x,y) \to 0$, the right-hand side tends to zero. Therefore by the squeeze theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

This doesn't depend on the path by which (x,y) might approach $(0,0)$.

End of Example 125. □

As a practical matter, if we know that f is continuous at a point (a,b), then we can evaluate the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

by simply plugging in $x=a$ and $y=b$. For example,

Example 126 (Polynomial functions)**.** Consider the polynomial

$$f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y.$$

Suppose we want to know

$$\lim_{(x,y)\to(1,2)} f(x,y).$$

Fact: polynomials are always continuous.

Since f is a polynomial (of two variables, x and y), it is continuous. Therefore, we can just plug in $x = 1$ and $y = 2$ to find the limit:

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 8 - 4 + 3 + 4 = 11.$$

Polynomials of multiple variables are a deep an interesting field of study.

End of Example 126. □

Fact: We can combine continuous functions together (e.g., by adding/subtracting them, multipying/dividing them, and by function composition), and the resulting function will be continuous on its domain.

Example 127 (Rational functions)**.** Consider the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

This is an example of a ***rational function***, which is a quotient of two polynomials. At what points in the xy-plane is the function f continuous?

- f is a rational function, and therefore it is continuous at all points on its domain, which is

$$D = \left\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\right\}$$

- f is not defined at $(0,0)$ and is therefore not continuous at that point.

End of Example 127. □