Math 243: Homework 03

Due Fridy, September 11

Problem 1. Consider

$$\mathbf{r}(t) = (\cos(2t), \sin(2t)), \qquad 0 \le t \le 2\pi.$$

This parameterization traces out the unit circle, whose circumference is 2π, but $\int_0^{2\pi} |\mathbf{r}'(t)|\, dt = 4\pi$. Explain why there is no contradiction. What does the integral measure in this example?

Problem 2. Consider $\mathbf{r}(t) = (t^3, t^2)$.

(a) Compute the velocity at time $t = 0$. Why does the direction formula $\mathbf{r}'(t)/|\mathbf{r}'(t)|$ fail at $t = 0$?

(b) Convert the curve to Cartesian form by eliminating the variable t and sketch the curve. Explain geometrically what goes wrong.

Problem 3 (Converting parametric equations to Cartesian form)**.** Consider the curve given by the parametric equations

$$x(t) = \frac{t-2}{t+1}, \quad y(t) = \frac{t}{t-1}$$

where $-\frac{1}{2} \le t \le \frac{1}{2}$.

(a) Find a Cartesian equation for the curve. Sketch a graph, and show the direction of travel. *(Hint: be careful about precisely which values of x your equation holds for.)*

(b) How does your answer change if we specify $-1 < t < 1$? Plot the graph in that case as well. *(Note that we can't take $t = -1$ or $t = 1$ because that would result in divison by zero!)*

Problem 4 (Arc length of parameterized curve)**.** Compute the arc length of the parameterized curve:

$$\begin{cases} x(t) = \frac{4}{3}t^3 + t - 2 \\ y(t) = \frac{1}{27}\left(9t^2 + 2\right)^{\frac{3}{2}} \end{cases}$$

for $0 \le t \le 1$. (Don't forget to use the chain rule!)

Problem 5 (Arc length - 3d)**.** Find the length of the following curve in 3d space:

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ t^2 \\ t^3 \end{bmatrix}, \quad 0 \le t \le 9.$$

Problem 6 (Parametric equations - tangent lines)**.** Let C be the curve defined by the parametric equations

$$\begin{cases} x(t) = t^3 - 12t \\ y(t) = 10 - t^2 \end{cases} \tag{1}$$

for $0 \le t \le 3$.

Here's a plot of the graph of this curve: `https://www.desmos.com/calculator/bvti5ck5r0`

(a.) Find an equation for the tangent line to C at $t = 1$. Your final answer should be an equation of a line in *slope-intercept form:* $y = mx + b$.

(b.) What is the equation of the normal line to C at $t = 1$?

(c.) Sketch the curve, the tangent line, and the normal line. (Please label them).

(d.) Compute the value of $\frac{d^2y}{dx^2}$ at the point $t = 1$.

(e.) Based on your answer in part (d.), is the curve C concave up or concave down at $t = 1$? *(Note: You can get the answer by looking at the graph, but the point of this part is that you know how to interpret part (d.)*

Problem 7 (Differentiating parametric equations)**.** Consider the curve C given by the parametric equations

$$\begin{cases} x = \frac{t}{1+t} \\ y = \sqrt{1+t} \end{cases}$$

for $0 \leq t < \infty$.

(a.) Compute $\frac{dy}{dx}$.

(b.) Let $f(t) = \frac{dy}{dx}(t)$. In your words, provide a geometric interpretation or explanation of the function f.

(c.) Compute the second derivative $\frac{d^2y}{dx^2}$.

(d.) Based on your answer in part (c.), at the point $t = 1$, is the curve C concave up, concave down, or neither? Justify your answer.